**Math 318 Fall 2008**
**Exercises Due: October 31**

3.4.12 For typographical reasons, let $\overrightarrow{\mathbf{x}}$ denote $\begin{bmatrix} x \\ y \end{bmatrix}$. Let $F\left(\overrightarrow{\mathbf{x}}\right) = x^2+4\,x+y+y^3-14$. Let $\mathcal{C} = \left\{\overrightarrow{\mathbf{x}} \in \mathbb{R}^2 : F\left(\overrightarrow{\mathbf{x}}\right) = 0\right\}$. Notice that the point $\overrightarrow{\mathbf{p}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ belongs to $\mathcal{C}$. The curve $\mathcal{C}$ defines $y$ as a function $f$ of $x$ in a neighborhood of $\overrightarrow{\mathbf{p}}$. That is, there is a disk $B_\delta\left(\overrightarrow{\mathbf{p}}\right)$ such that $B_\delta\left(\overrightarrow{\mathbf{p}}\right) \cap \mathcal{C}$ is the graph of a function $f$ defined on an open interval centered at 2. The purpose of this exercise is to calculate the order 3 Taylor polynomial

$$P_{f,2,3}(x) = f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}\cdot(x-2)^2 + \frac{f'''(2)}{3!}\cdot(x-2)^3$$

without explicitly calculating $f(x)$. Since $\overrightarrow{\mathbf{p}} = \begin{bmatrix} 2 \\ f(2) \end{bmatrix}$ we know that $f(2) = 1$. Use the rule of de Sluse, namely

$$f'(x) = -\frac{D_1(F)\left(\begin{bmatrix} x \\ f(x) \end{bmatrix}\right)}{D_2(F)\left(\begin{bmatrix} x \\ f(x) \end{bmatrix}\right)},$$

to calculate $\kappa = f'(2)$. The Talor polynomial can then be written as

$$P_{f,2,3}(x) = 1 + \kappa\cdot(x-2) + a\cdot(x-2)^2 + b\cdot(x-2)^3$$

where $\kappa$ has been calculated but $a$ and $b$ are as yet undetermined. Our goal is to calculate $a$ and $b$. In essence, we will calculate $f''(2)$ and $f'''(2)$ without even knowing what $f(x)$ is. To do that, let $u = x-2$, or equivalently $x = u+2$. Then

$$P_{f,2,3}(x) = P_{f,2,3}(u+2) = 1 + \kappa\cdot u + a\cdot u^2 + b\cdot u^3.$$

For $u$ sufficiently small, the point $\begin{bmatrix} u+2 \\ f(u+2) \end{bmatrix}$ lies on $\mathcal{C}$: $F\left(\begin{bmatrix} u+2 \\ f(u+2) \end{bmatrix}\right) = 0$. If we replace $f(u+2)$ with the approximation $P_{f,2,3}(u+2)$, then we will not get a point exactly on $\mathcal{C}$ but the error will involve only powers of $u$ that are greater than 3. That is, $F\left(\begin{bmatrix} u+2 \\ P_{f,2,3}(u+2) \end{bmatrix}\right) = E(u)$ where $E$ is a polynomial in $u$ that has no terms of degree 3 or less. Calculate the coefficients of $u^2$ and $u^3$ in $F\left(\begin{bmatrix} u+2 \\ P_{f,2,3}(u+2) \end{bmatrix}\right)$. Set these coefficients equal to 0 and use the equations to determine $a$ and $b$.

3.4.13 For typographical reasons, let $\overrightarrow{\mathbf{x}}$ denote $\begin{bmatrix} x \\ y \end{bmatrix}$. Let $F\left(\begin{bmatrix} \overrightarrow{\mathbf{x}} \\ z \end{bmatrix}\right) = 2xz+2\,y+y^2z-6$. Let $\mathcal{S} = \left\{\begin{bmatrix} \overrightarrow{\mathbf{x}} \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\left[\right.\right.\right.$ Notice that the point $\overrightarrow{\mathbf{p}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \overrightarrow{\mathbf{a}} \\ 1 \end{bmatrix}$ belongs to $\mathcal{S}$. The surface $\mathcal{S}$ defines $z$ as a function $f$ of $x$ and $y$ in a neighborhood of $\overrightarrow{\mathbf{p}}$. That is, there is a ball $B_\delta\left(\overrightarrow{\mathbf{p}}\right)$ such that $B_\delta\left(\overrightarrow{\mathbf{p}}\right) \cap \mathcal{S}$ is the graph of a function $f$ defined on an open disk centered at $\overrightarrow{\mathbf{a}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The purpose of this exercise is to calculate the order 2 Taylor polynomial

$$\begin{aligned} P_{f,\overrightarrow{\mathbf{a}},2}\left(\overrightarrow{\mathbf{x}}\right) &= f\left(\overrightarrow{\mathbf{a}}\right) + D_1(f)\left(\overrightarrow{\mathbf{a}}\right)\cdot(x+1) + D_2(f)\left(\overrightarrow{\mathbf{a}}\right)\cdot(y-2) \\ &\quad + a\cdot(x+1)^2 + b\cdot(x+1)(y-2) + c\cdot(y-2)^2 \end{aligned}$$

without explicitly calculating $f(x)$. Since $\overrightarrow{\mathbf{p}} = \begin{bmatrix} \overrightarrow{\mathbf{a}} \\ f\left(\overrightarrow{\mathbf{a}}\right) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, we know that $f\left(\overrightarrow{\mathbf{a}}\right) = 1$. Use the rule of de Sluse to calculate $\lambda = D_1(f)\left(\overrightarrow{\mathbf{a}}\right)$ and $\mu = D_2(f)\left(\overrightarrow{\mathbf{a}}\right)$. The Talor polynomial can then be written as

$$P_{f,\vec{\mathbf{a}},2}\left(\vec{\mathbf{x}}\right) = 1 + \lambda \cdot (x+1) + \mu \cdot (y-2) + a \cdot (x+1)^2 + b \cdot (x+1)(y-2) + c \cdot (y-2)^2$$

where $\lambda$ and $\mu$ have been calculated but $a$, $b$, and $c$ are as yet undetermined. Letting $u = x+1$ and $v = y-2$, we have

$$P_{f,\vec{\mathbf{a}},2}\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right) = 1 + \lambda \cdot u + \mu \cdot v + a \cdot u^2 + b \cdot uv + c \cdot v^2.$$

Our goal is to calculate $a$, $b$, and $c$ without even knowing what $f(x,y)$ is. For $u$ and $v$ sufficiently small, the point $\begin{bmatrix} u-1 \\ v+2 \\ f\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right) \end{bmatrix}$ lies on $\mathcal{C}$: $F\left(\begin{bmatrix} u-1 \\ v+2 \\ f\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right) \end{bmatrix}\right) = 0$. If we replace $f\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right)$ with the approximation $P_{f,\vec{\mathbf{a}},2}\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right)$, then we will not get a point exactly on $\mathcal{C}$ but the error will involve only powers of $u$ that are greater than 3. That is, $F\left(\begin{bmatrix} u-1 \\ v+2 \\ P_{f,\vec{\mathbf{a}},2}\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right) \end{bmatrix}\right) = E(u,v)$ where $E$ is a polynomial in $u$ and $v$ that has no monomials of degree 2 or less. Calculate the coefficients of $u^2$, $uv$, and $v^2$ in $F\left(\begin{bmatrix} u-1 \\ v+2 \\ P_{f,\vec{\mathbf{a}},2}\left(\begin{bmatrix} u-1 \\ v+2 \end{bmatrix}\right) \end{bmatrix}\right)$. Set these coefficients equal to 0 and use the equations to determine $a$, $b$, and $c$.

**Formulas** Some formulas to spare you the busy work of 3.3.13. If $f(x,y) = \sqrt{x+y+xy}$, then

$$D_{(1,0)}(f)(x,y) = \frac{1+y}{2\sqrt{x+y+xy}} \quad \text{and} \quad D_{(0,1)}(f)(x,y) = \frac{1+x}{2\sqrt{x+y+xy}},$$

$$D_{(1,1)}(f)(x,y) = \frac{1}{4}\frac{(-1+x+y+xy)}{(x+y+xy)^{3/2}},$$

and

$$D_{(2,0)}(f)(x,y) = -\frac{(1+y)^2}{4(x+y+xy)^{3/2}} \quad \text{and} \quad D_{(0,2)}(f)(x,y) = -\frac{(1+x)^2}{4(x+y+xy)^{3/2}}.$$