Introduction to Real Analysis

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0.1 ORDERED FIELDS

In this section we will discuss the algebraic and order properties of $\mathbb{Q}$. The following notation for standard number systems will be employed: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$, $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$, $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$, $\mathbb{Z}^\times = \{\pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$, $\mathbb{Q} = \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{Z}^\times\}$, and $\mathbb{Q}^\times = \{m/n \mid m, n \in \mathbb{Z}^\times\}$. Once we introduce the real numbers, we will denote them by $\mathbb{R}$. Then $\mathbb{C} = \{x + y \cdot i : x, y \in \mathbb{R}\}$ will denote the complex numbers. The goal of this chapter is to understand the analytic property that sets the real number system apart from the rational number system.

Monoids, Semigroups, and Groups

A *binary operation* $\odot$ on a set S is a function $\odot : S \times S \to S$. The notation $s_1 \odot s_2$ is generally used instead of $\odot(s_1, s_2)$. If $s_1 \odot s_2 = s_2 \odot s_1$ for all $s_1, s_2 \in S$, then $\odot$ is said to be *commutative* or *abelian*. We often use the symbol $+$ to denote a commutative operation—obviously if we are simultaneously considering two different commutative operations, such as we do with ordinary arithmetic, we cannot use $+$ to denote both of them, even if they are both commutative. If $s_1 \odot (s_2 \odot s_3) = (s_1 \odot s_2) \odot s_3$ for all $s_1, s_2, s_3 \in S$, then $\odot$ is said to be *associative*. The ordinary arithmetic operations of addition $+$ and multiplication $\cdot$ are commutative and associative binary operations on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{Q}$. Multiplication is also a commutative, associative binary operation on $\mathbb{Z}^\times$ and $\mathbb{Q}^\times$, but addition is not an operation on these sets. If $\odot$ is an associative operation on S, then the ordered pair $(S, \odot)$ is said to be a *monoid*.

EXAMPLE 1 *The properties of associativity and commutativity are independent. Using a two element set $S = \{a, b\}$, show that a binary operation may be commutative without being associative or associative without being commutative.*

Solution: If a set X has a finite number n of elements, then we may record a binary operation on X by enumerating the elements as $x_1, x_2, \ldots, x_n$, forming an $(n+1) \times (n+1)$ matrix, placing the row vector $[\odot, x_1, x_2, \ldots, x_n]$ in the first row, its transpose in the first column, and placing the element $x_i \odot x_j$ in the $(i+1)^{\text{th}}$ row, $(j+1)^{\text{th}}$ column. Consider the three binary operations $\bigcirc$, $\star$, and $\bullet$, defined by the following tables:

$$\begin{array}{ccc} \bigcirc & a & b \\ a & b & b \\ b & b & a \end{array} \qquad \begin{array}{ccc} \star & a & b \\ a & a & a \\ b & b & b \end{array} \qquad \begin{array}{ccc} \bullet & a & b \\ a & a & b \\ b & b & a \end{array}.$$

Since $a \bigcirc b = b = b \bigcirc a$ we see that $\bigcirc$ is commutative. However,

$$a \bigcirc (a \bigcirc b) = a \bigcirc b = b \neq a = b \bigcirc b = (a \bigcirc a) \bigcirc b.$$

Thus, $\bigcirc$ is commutative but not associative.

Since $a \star b = a \neq b = b \star a$ we see that $\star$ is *not* commutative. Notice that if x, y are used to represent either element of S and $z = x \star y$ we have

$$a \star (x \star y) = a \star z = a = a \star y = (a \star x) \star y$$

and

$$b \star (x \star y) = b \star z = b = b \star y = (b \star x) \star y.$$

Thus, $\star$ is associative but not commutative.

Finally, for good measure we have thrown in the operation $\bullet$, which is both commutative and associative. ■

If $e \in S$ satisfies $s \odot e = e \odot s = s$ for every $s \in S$, then e is said to be an *identity element* for $\odot$. An identity element, if it exists, is unique: if $s' \odot e = s'$ for every $s' \in S$ and if $e' \odot s = s$ then, by setting $s' = e'$ and $s = e$, we have $e' = e' \odot e = e$. A monoid with an identity element is said to be a *semigroup*. When we use the symbol $+$ to denote the operation of a commutative semigroup, whether or not it is the familiar operation of elementary school addition, we use the symbol 0 to denote the identity element.

If $\odot$ is an operation on a set S, if $\odot$ has an identity element e, and if for a given element s of S there is an element t such that $s \odot t = t \odot s = e$, then s is said to be *invertible* and the element t is said to be *an inverse* of s. An invertible element s in a semigroup has a unique inverse: if $s \odot t = e$ and $u \odot s = e$ then

$$u = u \odot e = u \odot (s \odot t) = (u \odot s) \odot t = e \odot t = t.$$

Because of this uniqueness, we may use a special symbol, s^{-1}, to denote the inverse of an invertible element s. When we use the symbol $+$ to denote the operation of a commutative semigroup, whether or not it is the familiar operation of elementary school addition, we use the notation $-s$ to denote the inverse of s. Every semigroup $(S, \odot)$ has at least one invertible element: if e is the identity element, then the equation $e \odot e = e$ shows that e is invertible and $e^{-1} = e$. A semigroup may have no invertible element other than the identity—the semigroup $(\mathbb{Z}^{\times}, \cdot)$ is an example. On the other hand it may happen that every element of a semigroup $(G, \odot)$ is invertible; in this case we call $(G, \odot)$ a *group*. Examples of groups are $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{Q}^{\times}, \cdot)$.

Ordered Fields

A field $(\mathbb{F}, +, \cdot)$ is an ordered triple in which $\mathbb{F}$ is a set, $+$ is a binary operation on $\mathbb{F}$ called addition, and $\cdot$ is a binary operation on $\mathbb{F}$ called multiplication. The field operations on $\mathbb{F}$ must have the following properties

- $(\mathbb{F}, +)$ is a commutative group with identity element 0;
- $(\mathbb{F}, \cdot)$ is a commutative semigroup with identity element 1, which is not the identity element for addition;
- Every element x of $(\mathbb{F}, \cdot)$ except 0 is invertible—we write the multiplicative inverse of x as $1/x$;
- Multiplication distributes over addition: $s \cdot (t + u) = s \cdot t + s \cdot u$ for all $s, t, u \in \mathbb{F}$.

We usually refer to a field $(\mathbb{F}, +, \cdot)$ by just stating the set $\mathbb{F}$. If $a \in \mathbb{F}$, then there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$. In particular, there is an element -1. The next theorem tells us that some aspects of field arithmetic are completely familiar.

Theorem 1 *For all $a, b \in \mathbb{F}$ we have*

i) $a \cdot 0 = 0$,
ii) $(-1) \cdot a = -a$, and
iii) $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$,
iv) $(-a) \cdot (-a) = a \cdot a$.

Proof: We have $a = a \cdot 1 = a \cdot (1 + 0) = a \cdot 1 + a \cdot 0 = a + a \cdot 0$. Add $-a$ to the expressions at the ends of this chain of equalities:

$$0 = (-a) + a = (-a) + (a + a \cdot 0) = (-a + a) + a \cdot 0 = 0 + a \cdot 0 = a \cdot 0.$$

Thus, $a \cdot 0 = 0$ for all $a \in \mathbb{F}$, as required.

Next, we use the equation just obtained to see that

$$a + (-1) \cdot a = a \cdot 1 + a \cdot (-1) = a \cdot (1 + (-1)) = a \cdot 0 = 0.$$

It follows that $(-1) \cdot a = -a$ for all $a \in \mathbb{F}$.

Turning to the two equations of part (iii), we have

$$(-a) \cdot b = ((-1) \cdot a) \cdot b = (-1) \cdot (a \cdot b) = -(a \cdot b)$$

and

$$a \cdot (-b) = a \cdot ((-1) \cdot b) = (a \cdot (-1)) \cdot b = (-a) \cdot b.$$

Finally, $(-1) \cdot (-1) = -(-1) = 1$. Thus,

$$(-a) \cdot (-a) = ((-1) \cdot a) \cdot ((-1) \cdot a) = (-1) \cdot (-1) \cdot a \cdot a = a \cdot a.$$

■

Notice that, by definition, a field must have at least two elements, 0 and 1. There is a field $\mathbb{F}_2$ with exactly these two elements:

$$\begin{array}{ccc} + & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{ccc} \cdot & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

This example shows that it is possible that $1+1=0$ in a field. We can also construct a field such that $1+1\neq 0$ but $1+1+1=0$. If $\mathbb{F}$ is a field such that $\underbrace{1+1+\cdots+1}_{n}\neq 0$ for every positive integer n, then we say that $\mathbb{F}$ has *characteristic* 0. We then denote the element $\underbrace{1+1+\cdots+1}_{n}$ by n. In this way we obtain a copy of the integers $\mathbb{Z}$ in every field of characteristic 0.

If a field $\mathbb{F}$ has a subset P with the properties

- $a,b\in P \Rightarrow a+b\in P$, and
- $a,b\in P \Rightarrow a\cdot b\in P$, and
- $a\in\mathbb{F}\Rightarrow$ precisely one of the following three relations occurs: $a\in P$ or $a=0$ or $-a\in P$,

then P is said to be a *positive class*. We also say that $\mathbb{F}$ is ordered by P and that $\mathbb{F}$ is an ordered field. An element a of P is said to be *positive* and we write $a>0$ or $0<a$. If $-a\in P$ then we say that a is *negative* and write $0>a$ or $a<0$. If a and b are elements of $\mathbb{F}$ with $a-b>0$, then we write $a>b$ or $b<a$ and say that a is greater than b or b is less than a. If $a\neq b$ then $a<b$ or $b<a$. In general, for any $a,b\in\mathbb{F}$, not necessarily distinct, we have $a\leq b$ or $b\leq a$ where $x\leq y$ means $x<y$ or $x=y$. It is easy to show that if $a\leq b$ and $b\leq a$ then $a=b$. Also, if $a\leq b$ and $b\leq c$ then $a\leq c$.

Theorem 2 *An ordered field has characteristic 0.*

Proof: First let us observe that 1 is in the positive class. If it were not, then -1 would be in the positive class. It would follow that $1=(-1)\cdot(-1)$ would also be in the positive class, which is closed under multiplication. But the third property of positive classes tells us that 1 and -1 cannot both be in the positive class. The contradiction we have obtained implies that 1 must be in the positive class. It follows that $1+1$, and in general, $\underbrace{1+1+\cdots+1}_{n}$ is also in the positive class for any $n\in\mathbb{Z}^+$. In particular, $\underbrace{1+1+\cdots+1}_{n}\neq 0$. ■

Absolute Value

If $a\in\mathbb{F}$, then we define the *absolute value* $|a|$ of a by

$$|a|=\begin{cases} a & \text{if } 0<a \\ 0 & \text{if } a=0 \\ -a & \text{if } a<0 \end{cases}.$$

The inequality

$$||a|-|b||\leq|a\pm b|$$

for all $a,b\in\mathbb{F}$ is sometimes useful. The absolute value also satisfies the triangle *inequality*

$$|a\pm b|\leq|a|+|b|$$

for all $a,b\in\mathbb{F}$. If we define d by $d(x,y)=|x-y|$ for all $x,y\in\mathbb{F}$ then d satisfies

- $d(x,y)\geq 0$ for all $x,y\in\mathbb{F}$;

- $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{F}$;
- $d(x,y) = 0$ if and only if $x = y$;
- $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in \mathbb{F}$.

The last property is derived as follows:

$$d(x,z) = |x-z| = |(x-y)+(y-z)| \le |x-y| + |y-z| = d(x,y) + d(y,z).$$

0.2 THE REAL NUMBER FIELD

A metric space is a space in which we can measure distances between points. The distance functions that we will use are not necessarily the standard Euclidean distances with which we are familiar. Nevertheless, we will impose some reasonable conditions that a distance function must satisfy in order that it does not depart from our intuitive notions of distance.

Archimedean Fields

An ordered field $\mathbb{F}$ is *archimedean* if for every $x \in \mathbb{F}$ there is an $n \in \mathbb{Z}$ with $x < n$.

Theorem 3 *Suppose that $\mathbb{F}$ is an archimedean ordered field. Then for every $x \in \mathbb{F}$ with $x > 0$ there is an $n \in \mathbb{Z}$ with $1/n < x$.*

Proof: Since $x > 0$ it is not 0, hence invertible. Because $\mathbb{F}$ is archimedean, there is an n such that $1/x < n$. Therefore $1/n < x$. ■

Theorem 4 *The ordered field $\mathbb{Q}$ is archimedean.*

Proof: Let $x = m/\ell$ with $m \in \mathbb{Z}$, $\ell \in \mathbb{Z}^+$. By the division algorithm of number theory, there are unique integers q and r with $0 \le r < \ell$ such that $m = q\ell + r$. Then $m \le q\ell + \ell$, or $m/\ell \le q + 1$, or $m/\ell < n$ with $n = q + 2$.. ■

Definition 5 *If $\{a_n\}$ is a sequence with values in $\mathbb{Q}$ and if $x \in \mathbb{Q}$, then we say that $\{a_n\}$ converges to x and we write $a_n \to x$ if for every $\epsilon \in \mathbb{Q}^+$ there is an N such that $|x - a_n| \le \epsilon$ for all $n \ge N$. We also say that x is the limit of $\{a_n\}$ and write $\lim_{n\to\infty} a_n = x$.*

Definition 6 *If $\{a_n\}$ is a $\mathbb{Q}$-valued sequence, then we say that $\{a_n\}$ is a Cauchy sequence if for every $\epsilon \in \mathbb{Q}^+$ there is an N such that $|a_m - a_n| \le \epsilon$ for all $m, n \ge N$.*

The Real Numbers

In this section we will sketch (without proofs) a construction of the real number field $\mathbb{R}$. Let R denote the set of all $\mathbb{Q}$-valued Cauchy sequences. If $x = \{x_n\}$ and $y = \{y_n\}$ belong to R we write $x \sim y$ if $\lim_{n\to\infty} |x_n - y_n| = 0$. This is an equivalence relation. We let $\mathbb{R}$ denote the set of equivalence classes of R—in the notation of set theory, $\mathbb{R} = R/\sim$. Let $[r]$ denote the equivalence class of an element $r = \{r_n\}$ of R. We define an addition on $\mathbb{R}$ by $[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$. To show that this is really an operation from $\mathbb{R} \times R \to R$ we must verify that (i) $\{x_n + y_n\}$ is a Cauchy sequence if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, and (ii) $\{x_n + y_n\} \sim \{x'_n + y'_n\}$ if $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$. (Obviously the addition of equivalence classes would not be a well defined operation if the result depended on the representatives chosen to carry out the operation.) We define a multiplication on $\mathbb{R}$ by $[\{x_n\}] \cdot [\{y_n\}] = [\{x_n y_n\}]$. To show that this is really an operation from $\mathbb{R} \times R \to R$ we must verify that (i) $\{x_n y_n\}$ is a Cauchy sequence if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, and (ii) $\{x_n y_n\} \sim \{x'_n y'_n\}$ if $\{x_n\} \sim \{x'_n\}$ and

$\{y_n\} \sim \{y'_n\}$. (Again, the result of an operation on equivalence classes would not be well defined if it depended on the representatives chosen to carry out the operation.) It can be shown that $\mathbb{R}$ is a field with these operations.

Let P be the subset of $\mathbb{R}$ consisting of all $[\{x_n\}]$ with the property that there exists an $\epsilon \in \mathbb{Q}^+$ and an N such that $x_n \geq \epsilon$ for all $n \geq N$. (To show that this is a well-defined set, it must be shown that if $\{x_n\}$ has this property and if $\{x'_n\} \sim \{x_n\}$ then there exists an ϵ' and an N' such that $x'_n \geq \epsilon'$ for all $n \geq N'$.) It may be shown that P is a positive class for $\mathbb{R}$. Because $\mathbb{R}$ is an ordered field, it contains a copy of $\mathbb{Q}$. It is easy to see that if $a \in \mathbb{Q}$, then the equivalence class $[\{a\}]$ containing the constant sequence $a, a, a, \ldots$ is the element of $\mathbb{R}$ that acts as the rational number a in $\mathbb{R}$. It may be shown that $\mathbb{Q}$ is *dense* in $\mathbb{R}$: every open interval (r_1, r_2) in $\mathbb{R}$ contains a rational number $[\{a\}]$ $(a \in \mathbb{Q})$. That is, if r_1 and r_2 are real numbers with $r_1 < r_2$, then there is a rational number a such that $r_1 < a < r_2$.

The real numbers are a genuine extension of the rational numbers. For example, we cannot solve the equation $x^2 = 2$ in $\mathbb{Q}$, but we can in $\mathbb{R}$. To see this, let $s_0 = d_0 = 1$ and, for $n \geq 1$, let $s_n = s_{n-1} + d_{n-1}$, and $d_n = 2s_{n-1} + d_{n-1}$. An argument by mathematical induction shows that $d_n^2 = 2s_n^2 + (-1)^{n+1}$. Also $s_n > n$ for all n. It follows that $\lim_{n\to\infty} d_n^2/s_n^2 = 2$. It can be shown that $x = \{d_n/s_n\}$ is a Cauchy sequence and $x^2 \sim \{2\}$.

The construction of the real numbers that has been given in outline does not appear natural to all eyes. As a result, other constructions have their advocates. Though the alternative approaches are quite different, they produce complete ordered fields that differ only cosmetically from our version of $\mathbb{R}$. Indeed, it can be shown that if $\mathbb{F}$ is a complete ordered field, then there is an invertible function φ from $\mathbb{F}$ onto $\mathbb{R}$ that preserves order (i.e., $\varphi(a) < \varphi(b)$ in $\mathbb{R}$ if $a < b$ in $\mathbb{F}$), distance (i.e., $|\varphi(a) - \varphi(b)| = |a - b|$ for all a, b in $\mathbb{F}$), and the arithmetic operations (i.e. $\varphi(a+b) = \varphi(a) + \varphi(b)$, and $\varphi(ab) = \varphi(a)\varphi(b)$ for all a, b in $\mathbb{F}$).

Complex Numbers

We can provide $\mathbb{R} \times \mathbb{R}$ with field operations by defining $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. The identity element for $+$ is $(0, 0)$ and the identity element for $\cdot$ is $(1, 0)$. If $(a, b) \neq (0, 0)$ then $(a, b)^{-1} = \left(a/\left(a^2 + b^2\right), -b/\left(a^2 + b^2\right)\right)$. The other field properties are easily verified. We denote the field $(\mathbb{R} \times \mathbb{R}, +, \cdot)$ by $\mathbb{C}$ and say that its elements are complex numbers. Notice that $(a, 0) + (c, 0) = (a + c, 0)$ and $(a, 0) \cdot (c, 0) = (ac, 0)$. In other words, $\mathbb{R} \times \{0\}$ is a copy of $\mathbb{R}$ in $\mathbb{C}$. We obtain additional algebraic structure by regarding $\mathbb{R} \times \mathbb{R}$ as a two-dimensional vector space over $\mathbb{R}$. Convenient basis vectors are $(1, 0)$ and $(0, 1)$. The first of these is the multiplicative identity 1. It is common to denote the other basis element, namely $(0, 1)$, by the symbol i. Then $(x, y) = x(1, 0) + y(0, 1) = x + yi$ for $x, y \in \mathbb{R}$. Notice that

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1(1, 0) = -1.$$

An element in $\mathbb{C}$ is said to be a *complex number.* An element of the form yi with $y \in \mathbb{R}$ is said to be *purely imaginary.* It is ironic that the terms "complex" and "imaginary" are associated with $\mathbb{C}$. Remember that $\mathbb{N}$ consists of equivalence classes of equinumerous sets, that $\mathbb{Z}$ consists of equivalence classes of elements of $\mathbb{N} \times \mathbb{N}$, that $\mathbb{Q}$ consists of equivalence classes of elements of $\mathbb{Z} \times \mathbb{Z}$, and that $\mathbb{R}$ consists of equivalence classes of Cauchy sequences in $\mathbb{Q}$. By contrast, the complex number system can be constructed without recourse to equivalence classes.

0.3 SEQUENCES IN METRIC SPACES

A metric space is a space in which we can measure distances between points. The distance functions that we will use are not necessarily the standard Euclidean distances with which we are familiar. Nevertheless, we will impose some reasonable conditions that a distance function must satisfy in order that it does not depart from our intuitive notions of distance.

Metric Spaces

If X is a set and d is a *nonnegative* function on $X \times X$, then d is called a *metric* or *distance function* and the ordered pair (X, d) is called a *metric space* if

- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, x) = 0$ for all $x \in X$;

- $d(x, y) > 0$ for all $x, y \in X$ with $x \neq y$; and
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the *triangle inequality*).

The following definitions apply to sequences with values in a metric space.

Definition 7 *If $\{a_n\}$ is a sequence with values in a metric space (X, d) and if $x \in X$, then we say that $\{a_n\}$ converges to x and we write $a_n \to x$ if for every $\epsilon > 0$ there is an N such that $d(a_n, x) \leq \epsilon$ for all $n \geq N$. We also say that x is the limit of $\{a_n\}$ and write $\lim_{n\to\infty} a_n = x$.*

Definition 8 *If $\{a_n\}$ is a sequence with values in a metric space (X, d), then we say that $\{a_n\}$ is a Cauchy sequence if for every $\epsilon > 0$ there is an N such that $d(a_m, a_n) \leq \epsilon$ for all $m, n \geq N$.*

Theorem 9 *A convergent sequence in a metric space is a Cauchy sequence.*

Proof: Suppose that (X, d) is a metric space, that $\{a_n\} \subset X$, that $x \in X$, and that $a_n \to x$. Let ϵ be any positive number. There exists an N such that $d(a_n, x) < \epsilon/2$ for all $n \geq N$. Then, if $m, n \geq N$, we have

$$d(a_m, a_n) \leq d(a_m, x) + d(x, a_n) \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

■

If $n_1, n_2, n_3, \ldots, n_k, \ldots$ is an increasing, infinite sequence of positive integers, then we say that $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$. For example, if $a_n = 1/n$ for $n \geq 1$ and $n_k = 2^k$ for $k \geq 1$, we see that $1/2, 1/4, 1/8, \ldots$ is a subsequence of $1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, \ldots$.

Definition 10 *If $\{a_n\}$ is a sequence with values in a metric space (X, d) and if $x \in X$, then we say that x is a subsequential limit of $\{a_n\}$ if there is a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = x$.*

A sequence need not have any subsequential limit, as the example $\{n\}_{n=1}^{\infty}$ shows. On the other hand, a sequence may have more than one subsequential limit. For example, if $\{a_n\}$ is the sequence in $\mathbb{Q}$ defined by $a_n = (-1)^n + 1/n$, then $\lim_{k\to\infty} a_{2k} = 1$ and $\lim_{k\to\infty} a_{2k+1} = -1$. The sequence

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 6, 7, 8, 1, 2, 3 \ldots$$

has every member of $\mathbb{Z}^+$ as a subsequential limit. Readers familiar with Cantor's "diagonalization process" will have no trouble producing a sequence in $\mathbb{Q}^+$ that has every member of $\mathbb{Q}^+$ as a subsequential limit. By contrast, a Cauchy sequence can have at most one subsequential limit, as the next theorem shows.

Theorem 11 *If (X, d) is a metric space, if $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence with values in X, if $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ that converges to $x \in X$, then $\{a_n\}_{n=1}^{\infty}$ converges to x.*

Proof: Let $\epsilon > 0$. There is a K such that $d(a_{n_k}, x) \leq \epsilon/2$ for $k \geq K$. There is an N such that $d(a_n, a_m) \leq \epsilon/2$ for $n, m \geq N$. Set $M = \max(n_K, N)$. Then for $m \geq M$ we have

$$d(a_m, x) \leq d(a_m, a_{n_K}) + d(a_{n_K}, x) \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

■

Cauchy sequences are sequences that "try" to converge. What sometimes prevents a Cauchy sequence from converging is the absence of a point at the location the sequence approaches—a "hole," so to speak, in the ambient space. For future reference we mention that there is a term for metric spaces that have no holes.

Definition 12 *A metric space in which every Cauchy sequence converges is said to be complete.*

Ordered Fields as Metric Spaces

If $\mathbb{F}$ is an ordered field with absolute value $|\cdot|$ and if $d(x,y) = |x-y|$ for $x, y \in \mathbb{F}$, then $(\mathbb{F}, d)$ is a metric space. The following definitions apply to sequences with values in an ordered field.

Definition 13 *If $\{a_n\}$ is a sequence with values in an ordered field $\mathbb{F}$, we say that:*

- $\{a_n\}$ is *bounded above* if there is a $U \in \mathbb{F}$ such that $a_n \leq U$ for all n;
  $\{a_n\}$ is *bounded below* if there is an $L \in \mathbb{F}$ such that $L \leq a_n$ for all n;
  $\{a_n\}$ is *bounded* if there is a $B \in \mathbb{F}$ such that $|a_n| \leq B$ for all n;
  $\{a_n\}$ is *strictly increasing* if $a_n < a_{n+1}$ for all n;
  $\{a_n\}$ is *nondecreasing* if $a_n \leq a_{n+1}$ for all n;
  $\{a_n\}$ is *strictly decreasing* if $a_n > a_{n+1}$ for all n;
  $\{a_n\}$ is *nonincreasing* if $a_n \geq a_{n+1}$ for all n;
  $\{a_n\}$ is *montone* if it is strictly increasing, nondecreasing, strictly decreasing, or nonincreasing.

Theorem 14 *A Cauchy sequence in an ordered field is bounded.*

Proof: Let $\{a_n\}$ be a.Cauchy sequence in an ordered field. Set $\epsilon = 1$ in the definition of a Cauchy sequence. Then $|a_n - a_N| \leq 1$ for all $n \geq N$. Thus, $-1 \leq a_n - a_N \leq 1$ for all $n \geq N$, or $a_N - 1 \leq a_n \leq a_N + 1$ for all $n \geq N$. It follows that

$$\min(a_1, a_2, \ldots, a_{N-1}, a_N - 1) \leq a_n \leq \max(a_1, a_2, \ldots, a_{N-1}, a_N + 1) \text{ for all } n \geq N.$$

■

Theorem 15 *A nondecreasing sequence that is bounded above in an ordered field is a Cauchy sequence. A nonincreasing sequence that is bounded below in an ordered field is a Cauchy sequence.*

Proof: Suppose that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing and $a_n < U$ for all n. Let $\delta = U - a_1 > 0$. If there is a tail $\{a_n\}_{n=N}^{\infty}$ and a value U^* such that $a_n = U^*$ for all $n \geq N$, then $|a_n - a_m| = 0$ for all $m, n \geq N$ and $\{a_n\}$ is Cauchy. Thus, we may suppose that $\{a_n\}_{n=1}^{\infty}$ does not have a constant tail. In particular, each tail of $\{a_n\}$ has infinitely many values. Consider $L_1 = [a_1, (U + a_1)/2)$ and $R_1 = [(U + a_1)/2, U)$. The tail of $\{a_n\}$ is in one of these intervals. Call it $I_1 = [\ell_1, r_1)$ and subdivide: $I_1 = [\ell_1, (\ell_1 + r_1)/2) \cup [(\ell_1 + r_1)/2, r_1)$. The tail of $\{a_n\}$ is in one of these intervals. Call it $I_2 = [\ell_2, r_2)$ and subdivide. Continue this process. Notice that the length of I_n is $\delta/2^n$. Given $\epsilon > 0$ there is an M such that $\delta/2^M \leq \epsilon$. Let N be the first element of $\{a_n\}$ in I_M. Then for all $m, n \geq N$ we have $|a_n - a_m| \leq r_M - \ell_M = \delta/2^M \leq \epsilon$. ■

Theorem 16 *If $\{a_n\}_{n=1}^{\infty}$ is a monotone sequence with values in an ordered field $\mathbb{F}$, if $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ that converges to $x \in \mathbb{F}$, then $\{a_n\}_{n=1}^{\infty}$ converges to x.*

Proof: Let us suppose that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing—the nonincreasing case is handled similarly. Let $\epsilon > 0$. There is a K such that $0 < x - a_{n_k} \leq \epsilon$ for $k \geq K$. Thus, for $n \geq n_K$ we have

$$0 < x - a_n < x - a_{n_k} \leq \epsilon.$$

■

Theorem 17 *If $\{a_n\}_{n=1}^{\infty}$ is a sequence with values in an ordered field $\mathbb{F}$, then $\{a_n\}$ contains a monotone subsequence.*

Proof: Let $\mathcal{I} = \{n : a_m \leq a_n \text{ for all } m \geq n\}$. If $\mathcal{I} = \{n_1, n_2, n_3, \ldots\}$ is infinite, then $\{a_{n_k}\}$ is a nonincreasing infinite subsequence of $\{a_n\}$. If $\mathcal{I} = \{n_1, n_2, n_3, \ldots, n_k\}$ is finite, then set $m_1 = n_k + 1$. Since $m_1 \notin \mathcal{I}$, there is a member of the sequence a_{m_2} with $m_1 < m_2$ and $a_{m_1} < a_{m_2}$. Since $m_2 \notin \mathcal{I}$, there is a member of the sequence a_{m_3} with $m_2 < m_3$ and $a_{m_4} < a_{m_3}$. Continuing in this way we obtain an infinite, increasing sequence. ■

Upper and Lower Bounds

Definition 18 *If S is a subset of an ordered field $\mathbb{F}$, then we say that S is bounded above if there is a $U \in \mathbb{F}$ such that $s \leq U$ for all $s \in S$, we say that S is bounded below if there is an $L \in \mathbb{F}$ such that $L \leq s$ for all $s \in S$, and we say that S is bounded if there is a $B \in \mathbb{F}$ such that $|s| \leq B$ for all $s \in S$.*

Definition 19 *Suppose that S is a subset of an ordered field $\mathbb{F}$ that is bounded above. If $U^* \in \mathbb{F}$ is an upper bound of S and if $U^* \leq U$ for every upper bound of S, then we say that U^* is a least upper bound for S in $\mathbb{F}$. Another term that is used is supremum. We write $U^* = \sup(S)$.*

A supremum of a subset S of $\mathbb{F}$ is not required to belong to S. That is, a supremum of S is not necessarily a maximum value of S: it is possible for $s < \sup(S)$ for *every* $s \in S$. For example, if $S = \{a \in \mathbb{Q} \,|\, a < 1\}$, then $\sup(S) = 1 \notin S$. However, the supremum of a subset of $\mathbb{F}$ must lie in $\mathbb{F}$. As a result, not every set with an upper bound has a least upper bound, as the next example will show.

EXAMPLE 2 *Consider the bounded subset $S = \left\{a \in \mathbb{Q} \,|\, a^2 < 2\right\}$ of $\mathbb{Q}$. Show that S has no least upper bound in $\mathbb{Q}$.*

Solution: Since $(5/4)^2 = 25/16 < 32/16 = 2$, we see that $5/4 \in S$. Therefore any upper bound is at least $5/4$. Certainly, then, any proposed upped bound x of S must satisfy $2/3 < x^2$. For any $x \in S$ with $x^2 > 2/3$, we will produce a $y \in S$ with $x < y$. This will show that no possible upper bound x can have $x^2 < 2$. Of course, no element of $\mathbb{Q}$ satisfies $x^2 = 2$. The proof will be complete by showing that if $x^2 > 2$, then we can find an upper bound y of S with $y^2 > 2$ and $y < x$. (The upshot of this argument is that, since a least upper bound of S in $\mathbb{Q}$ cannot satisfy any of the relations $x^2 < 2$, $x^2 = 2$, $x^2 > 2$, a least upper bound of S cannot exist in $\mathbb{Q}$.)

Let $f(u) = (3u+2)^2 / (16u)$. We calculate and $f'(u) = \left(9u^2 - 4\right) / \left(16u^2\right) > 0$ for $u > 2/3$. Thus, $f(u) < f(2)$ if $2/3 < u < 2$. Suppose that $x \in \mathbb{Q}$ satisfies $x^2 < 2$. Set $y = \left(3x^2 + 2\right) / (4x)$. Then

$$x = \frac{4x^2}{4x} = \frac{3x^2 + x^2}{4x} < \frac{3x^2 + 2}{4x} = y$$

and

$$y^2 = \left(\frac{3x^2+2}{4x}\right)^2 = \frac{\left(3x^2+2\right)^2}{16x^2} = f\left(x^2\right) < f(2) = 2.$$

Next, suppose that $x \in \mathbb{Q}$ satisfies $2 < x^2$. Set $y = \left(3x^2 + 2\right) / (4x)$. Then

$$y = \frac{3x^2 + 2}{4x} < \frac{3x^2 + x^2}{4x} = \frac{4x^2}{4x} = x$$

and

$$y^2 = \left(\frac{3x^2+2}{4x}\right)^2 = \frac{\left(3x^2+2\right)^2}{16x^2} = f\left(x^2\right) > f(2) = 2.$$

■

Definition 20 *Suppose that S is a subset of an ordered field $\mathbb{F}$ that is bounded below. If L^* is a lower bound of S and if $L \leq L^*$ for every lower bound of S, then we say that L^* is a greatest lower bound for S. Another term that is used is infimum. We write $U^* = \inf(S)$.*

If S is a subset of an ordered field $\mathbb{F}$, then let $-S = \{-s \ : \ s \in S\}$. Then U is an upper bound for S if and only if $-U$ is a lower bound for $-S$. Similarly, U^* is the supremum of S if and only if $-U^*$ is the infimum of $-S$. As a result of this observation, assertions about infima can often be deduced from the corresponding assertions about suprema. The next theorem illustrates this point.

Theorem 21 *Suppose that U^* is the supremum of a subset S of an ordered field $\mathbb{F}$. Then there is a nondecreasing sequence $\{a_n\} \subset S$ such that $a_n \to U^*$. Analogously, if L^* is the infimum of a subset S of an ordered field $\mathbb{F}$. Then there is a nonincreasing sequence $\{a_n\} \subset S$ such that $a_n \to L^*$.*

Proof: If $U^* \in S$ then we may set $a_n = U^*$ for all n. We may therefore assume $U^* \notin S$. Since $T_1 = U^* - 1$ is less than U^*, it is not an upper bound for S. As a result, there is an $a_1 \in S$ such that $T_1 < a_1 < U^*$. Let $T_2 = \max(a_1, U^* - 1/2)$. Since $T_2 < U^*$, it is not an upper bound for S. As a result, there is an $a_2 \in S$ such that $T_2 < a_2 < U^*$. Let $T_3 = \max(a_2, U^* - 1/3)$, find an $a_3 \in S$ such that $T_3 < a_3 < U^*$, and continue in this way. Given $\epsilon > 0$, let N be such that $1/N < \epsilon$. The for $n \geq N$ we have $0 < U^* - a_n < 1/N < \epsilon$. Thus, $a_n \to U^*$.

Next assume that L^* is the infimum of a subset S of an ordered field $\mathbb{F}$. Then $-L^*$ is the supremum of $-S$. By what we have shown, there is a sequence $\{b_n\} \subset (-S)$ such that $b_n \to (-L^*)$. Then, if $a_n = -b_n$, we have $\{a_n\} \subset S$ and $a_n \to L^*$. ■

0.4 COMPLETENESS PROPERTIES FOR ORDERED FIELDS

Throughout this section we assume that $\mathbb{F}$ is an ordered field. We will state five properties for $\mathbb{F}$ that are equivalent: if $\mathbb{F}$ has one of these properties, then it has all of them. We will sketch a construction of an ordered field $\mathbb{R}$ that does enjoy the five properties. By contrast, the subfield $\mathbb{Q}$ of $\mathbb{R}$ enjoys none of these properties.

Characterizations of Completeness

In Section 2 we observed that the rational numbers have gaps. The set $S_0 = \{a \in \mathbb{Q} : \ a^2 < 2\}$ is bounded above but has no least upper bound. If we let $S = \{s \in \mathbb{Q} : s \leq s_0 \text{ for some } s_0 \in S_0\}$ and $T = \{t \in \mathbb{Q} : s \leq t \text{ for all } s \in S\}$ then $\mathbb{Q} = S \cup T$ but there is, nevertheless, a gap where S ends and T begins: S has no maximum element and T has no minimum element. In effect there is a "hole" betwen S and T. There are a number of ways to describe an ordered field $\mathbb{F}$ that, unlike $\mathbb{Q}$, is free from "holes".

Theorem 22 *Let $\mathbb{F}$ be an ordered field. The following properties are equivalent.*

- **(i)** Every nonempty subset S of $\mathbb{F}$ that is bounded above has a supremum;
- **(i')** Every nonempty subset S of $\mathbb{F}$ that is bounded below has an infimum;
- **(ii)** Every nondecreasing sequence in $\mathbb{F}$ that is bounded above is convergent;
- **(ii')** Every nonincreasing sequence in $\mathbb{F}$ that is bounded below is convergent;
- **(iii)** Every Cauchy sequence is convergent;
- **(iv)** Every sequence $\{I_n\}$ of nested (i.e. $I_{n+1} \subset I_n$) closed, bounded intervals has a nonempty intersection.
- **(v)** For every partition $\mathbb{F} = S \cup T$ into two nonempty sets with $s < t$ for all $s \in S$ and $t \in T$, there is either an element $\sigma \in S$ with $s \leq \sigma$ for all $s \in S$ or an element $\tau \in T$ with $\tau \leq t$ for all $t \in T$.

Proof: (i) $\Rightarrow$ (ii)

Suppose that $\{a_n\} \subset \mathbb{F}$ is bounded above and nondecreasing. Let U^* be its supremum. Then, by Theorem 6 of Section 2, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to U^*$. By Theorem 4 of Section 2, we also have $a_n \to U^*$. In particular, $\{a_n\}$ is convergent.

(ii) $\Rightarrow$ (iii)

Suppose that $\{a_n\} \subset \mathbb{F}$ is a Cauchy sequence. Then $\{a_n\}$ is bounded. Moreover, $\{a_n\}$ has a monotone subsequence $\{a_{n_k}\}$ (which is also bounded). By property (ii), there is an $x \in \mathbb{F}$ such that $a_{n_k} \to x$. Because $\{a_n\}$ is Cauchy, $a_n \to x$ as well.

(iii) $\Rightarrow$ (ii)

A nondecreasing sequence in $\mathbb{F}$ that is bounded above is Cauchy, and therefore convergent.

(iii) $\Rightarrow$ (iv)

Suppose that $I_n = [a_n, b_n]$ with $I_{n+1} \subset I_n$. Then $\{a_n\}$ is nondecreasing and bounded above by b_1. Also, $\{b_n\}$ is nonincreasing and bounded below by a_1. Therefore, there exist α and β such that $a_n \to \alpha$ and $b_n \to \beta$. It is easy to see that $\alpha \leq \beta$ and $[\alpha, \beta] \subset \bigcap_{n=1}^{\infty} I_n$.

(iv) $\Rightarrow$ (v)

Suppose that $\mathbb{F} = S \cup T$ with $s < t$ for all $s \in S$ and $t \in T$. Choose $s_1 \in S$ and $t_1 \in T$. Let $I_1 = [s_1, t_1]$. Let $\delta = t_1 - s_1 > 0$. Let $m_1 = (s_1 + t_1)/2$. Consider $L_1 = [s_1, m_1]$ and $R_1 = [m_1, t_1]$. Either $m_1 \in S$ or $m_1 \in T$. If $m_1 \in S$, then let $I_2 = R_1$. If $m_1 \in T$, then let $I_2 = L_1$. Subdivide I_2 in the same way and continue the process. There results a sequence of nested, closed, bounded intervals $\{I_n\}$ with $I_n = [s_n, t_n]$, s_n belonging to S and t_n belonging to T. Let $\gamma \in \bigcap_{n=1}^{\infty} I_n$. Either $\gamma \in S$ or $\gamma \in T$ (because every element is in S or T). Suppose that $\gamma \in S$. We are to show that $s \leq \gamma$ for all $s \in S$. If not, there is an $s' \in S$ with $s' > \gamma$. Let $\epsilon = s' - \gamma$. Choose N so that $\delta/2^N < \epsilon$. Then

$$t_N = s_N + \frac{\delta}{2^N} \leq \gamma + \frac{\delta}{2^N} < \gamma + \epsilon = s',$$

which is a contradiction. In the case that $\gamma \in T$, an analogous argument shows that $\gamma \leq t$ for all $t \in T$.

(v) $\Rightarrow$ (i)

Suppose that S_0 is bounded above. Let U be an upper bound for S_0. Let $S = \{s \in \mathbb{F} \mid \; s \leq s_0 \text{ for some } s_0 \in S_0\}$. Clearly U is also an upper bound for S. Therefore $s < U + 1$ for all $s \in S$. Let $T = \{t \in \mathbb{F} \mid s < t \text{ for all } s \in S\}$. Then T is nonempty since $U + 1 \in T$. If $x \in \mathbb{F} \setminus S$ then $x > s$ for all $s \in S$. Thus $x \in T$. Consequently, $\mathbb{F} = S \cup T$. By property (v), either S has a maximum element σ or T has a minimum element τ. Whichever is the case, denote the extreme value by γ. It is clear that γ is an upper bound for S_0. We will show that γ is the least upper bound of S_0. To that end, consider $x_n = \gamma - 1/n$ for every $n \in \mathbb{Z}^+$. If $\gamma \in T$, then since $x_n < \gamma$ and γ is the minimum element of T, we conclude that $x_n \notin T$. In other words, there is an element $r_n \in S$ such that $x_n < r_n$. From the definition of S, we deduce that there is an $s_n \in S_0$ such that $x_n < s_n$. Now if $\gamma' < \gamma$ choose n such that $\gamma' < \gamma - 1/n < \gamma$. Then $\gamma' < \gamma - 1/n = x_n < s_n$. This shows that no number less than γ can be an upper bound for S_0. Therefore γ is the least upper bound of S_0. ■

The Real Numbers

Since the properties of the preceding theorem are equivalent, an ordered field either has none of them or all of them. The rational number field $\mathbb{Q}$ falls into the first category but the real number field $\mathbb{R}$ has all the completeness properties.

The following theorem states a useful property of the real numbers. At this point it is an easy consequence of completeness.

Theorem 23 *(Bolzano-Weierstrass) Every bounded real-valued sequence has a convergent subsequence.*

Proof: Every real-valued sequence $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. If $\{a_n\}$ is bounded then $\{a_{n_k}\}$ is also bounded. Because $\mathbb{R}$ is complete, the monotone bounded sequence $\{a_{n_k}\}$ converges. ■

Archimedean Fields

An ordered field $\mathbb{F}$ is an *archimedian* field if for every $x \in \mathbb{F}$ there is an $n \in \mathbb{N}$ such that $x < n$. There are ordered fields that are not archimedean, but the next theorem shows that complete ordered fields are archimedean.

Theorem 24 *If $\mathbb{F}$ is a complete ordered field, then $\mathbb{F}$ is archimedean.*

Proof: Suppose that $\mathbb{F}$ is not archimedean. That is, suppose that there is an $x \in \mathbb{F}$ such that $x > n$ for every integer n. Then the sequence $\{n\}_{n=1}^{\infty}$ is increasing and bounded above by x, hence convergent. Let y be the element in such that $\lim_{n\to\infty} n = y$. Then there exists an N such that $|y - n| \leq 1/4$ for $n \geq N$. For these n we have

$$1 = |(n + 1 - y) + (y - n)| \leq |n + 1 - y| + |y - n| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

which is a contradiction. ■