Math 4111 Fall 2008
Exam 1 - VP Debate Edition

1. Let $a_1 = 1$ and $a_n = \sqrt{2a_{n-1}}$ for $n > 1$. Show that $\{a_n\}_{n=1}^{\infty}$ converges and calculate its limit.

2. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded sequences of positive numbers and that $\lim_{n\to\infty} x_n = x$. Show that
$$\limsup_{n\to\infty} x_n y_n = x \limsup_{n\to\infty} y_n.$$

3. The rational numbers in the interval $(0,1)$ are countably infinite. In other words, there is a bijection $x : \mathbb{Z}^+ \to \mathbb{Q} \cap (0,1)$. If we write x_n instead of $x(n)$, then we have "enumerated" $\mathbb{Q} \cap (0,1)$ as a sequence $\{x_n\}_{n=1}^{\infty}$. What is $\limsup_{n\to\infty} x_n$? (Justify your answer!)

4. Show that
$$\overline{(A \cup B)} = \overline{A} \cup \overline{B}$$
for any two subsets A and B of a metric space.

5. Suppose that $\lim_{n\to\infty} x_n = 0$ in a complete normed linear space (i.e. a Banach space). Show that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that
$$\lim_{K\to\infty} (x_{n_1} + x_{n_2} + \cdots + x_{n_K})$$
exists.