

**Math 4111 Fall 2008**  
**Exercises September 2**

1. Suppose that  $\{a_n\}_{n=1}^{\infty}$  converges to  $x$  and to  $y$  in a metric space. Prove that  $x = y$ .
2. Suppose that  $\{a_n\}_{n=1}^{\infty}$  converges to  $x$  in an ordered field. Show that  $\{|a_n|\}_{n=1}^{\infty}$  converges to  $|x|$ .
3. Let  $(X, d)$  be a metric space. Let  $\overline{X}$  denote the set of Cauchy sequences with values in  $X$ . If  $\overline{x} = \{x_n\}$  and  $\overline{y} = \{y_n\}$  belong to  $\overline{X}$ , define  $\overline{x} \sim \overline{y}$  if given  $\epsilon > 0$  there is an  $N$  such that  $d(x_n, y_n) \leq \epsilon$  for all  $n \geq N$ . Show that  $\sim$  is an equivalence relation.
4. Let  $\overline{x} = \{x_n\}$  and  $\overline{y} = \{y_n\}$  be Cauchy sequences in an ordered field. Define  $\overline{x} + \overline{y} = \{x_n + y_n\}$  and  $\overline{x} \cdot \overline{y} = \{x_n \cdot y_n\}$ . Show that  $\overline{x} + \overline{y}$  and  $\overline{x} \cdot \overline{y}$  are Cauchy sequences.
5. Let  $\overline{X}$  denote the set of Cauchy sequences with values in an ordered field. Let  $P$  be the subset of  $\overline{X}$  comprising all  $\overline{x} = \{x_n\} \in \overline{X}$  with the property that, given  $\epsilon > 0$  there exists an  $N$  such that  $x_n \geq \epsilon$  for all  $n \geq N$ . Suppose that  $\overline{x} = \{x_n\} \in P$  and  $\overline{y} = \{y_n\} \in P$ . Suppose that  $\overline{z} = \{z_n\}$  is equivalent to  $\overline{x}$  in the sense of Exercise 3. Show that  $\overline{x} + \overline{y}$ ,  $\overline{x} \cdot \overline{y}$ , and  $\overline{z}$  belong to  $P$ .
6. Let  $s_0 = d_0 = 1$ . For  $n \geq 1$  let  $s_n = s_{n-1} + d_{n-1}$ , and  $d_n = 2s_{n-1} + d_{n-1}$ . Use mathematical induction to show that  $d_n^2 = 2s_n^2 + (-1)^{n+1}$ . Prove that the sequence  $\{d_n/s_n\}$  of rational numbers is a Cauchy sequence. Prove that the sequence  $\{d_n^2/s_n^2\}$  of rational numbers converges to 2. (This last result shows that the Cauchy sequence  $\{d_n/s_n\}$  does not converge in the rationals. The rational numbers are therefore not complete.)