

**Math 4111 Fall 2008**  
**Exercises September 2**

1. Suppose that  $\{a_n\}_{n=1}^{\infty}$  converges to  $x$  and to  $y$  in a metric space. Prove that  $x = y$ .

**Solution** Suppose that  $\{a_n\}_{n=1}^{\infty}$  converges to  $x$  and to  $y$  in a metric space. Given  $\epsilon > 0$  there exists  $N_1$  such that  $d(a_n, x) \leq \epsilon/2$  for all  $n$  with  $n \geq N_1$  and there exists  $N_2$  such that  $d(a_n, y) \leq \epsilon/2$  for all  $n$  with  $n \geq N_2$ . Thus, if  $M = \max(N_1, N_2)$  we have

$$d(x, y) \leq d(x, a_M) + d(a_M, y) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The only nonnegative number that is less than or equal to any positive number  $\epsilon$  is 0. Therefore  $d(x, y) = 0$  and this implies  $x = y$ .

2. Suppose that  $\{a_n\}_{n=1}^{\infty}$  converges to  $x$  in an ordered field. Show that  $\{|a_n|\}_{n=1}^{\infty}$  converges to  $|x|$ .

**Solution** Given  $\epsilon > 0$  there exists an  $N$  such that  $|a_n - x| \leq \epsilon$  for all  $n$  with  $n \geq N$ . For these  $n$  we have

$$||a_n| - |x|| \leq |a_n - x| \leq \epsilon,$$

which proves that  $\lim_{n \rightarrow \infty} |a_n| = |x|$ .

3. Let  $(X, d)$  be a metric space. Let  $\overline{X}$  denote the set of Cauchy sequences with values in  $X$ . If  $\overline{x} = \{x_n\}$  and  $\overline{y} = \{y_n\}$  belong to  $\overline{X}$ , define  $\overline{x} \sim \overline{y}$  if given  $\epsilon > 0$  there is an  $N$  such that  $d(x_n, y_n) \leq \epsilon$  for all  $n \geq N$ . Show that  $\sim$  is an equivalence relation.

**Solution** Since  $d(x_n, x_n) = 0 \leq \epsilon$  for all  $n \geq N$  and every  $\epsilon > 0$  it is true that  $\{x_n\} \sim \{x_n\}$  for all  $\{x_n\} \in \overline{X}$ . Since  $d(x_n, y_n) = d(y_n, x_n)$  for all  $n$  it is clear that it is true that  $\{x_n\} \sim \{y_n\}$  implies that  $\{y_n\} \sim \{x_n\}$ . Finally if  $\{x_n\} \sim \{y_n\}$ , if  $\{y_n\} \sim \{z_n\}$ , and if  $\epsilon > 0$ , there are integers  $N_1$  and  $N_2$  such that  $d(x_n, y_n) \leq \epsilon/2$  for all  $n \geq N_1$  and  $d(y_n, z_n) \leq \epsilon/2$  for all  $n \geq N_2$ . Therefore, if  $n \geq \max(N_1, N_2)$  we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that  $\{x_n\} \sim \{y_n\}$  and  $\{y_n\} \sim \{z_n\}$  implies  $\{x_n\} \sim \{z_n\}$ . Thus, the relation  $\sim$  is reflexive, symmetric, and transitive, hence an equivalence.

4. Let  $\overline{x} = \{x_n\}$  and  $\overline{y} = \{y_n\}$  be Cauchy sequences in an ordered field. Define  $\overline{x} + \overline{y} = \{x_n + y_n\}$  and  $\overline{x} \cdot \overline{y} = \{x_n \cdot y_n\}$ . Show that  $\overline{x} + \overline{y}$  and  $\overline{x} \cdot \overline{y}$  are Cauchy sequences.

**Solution** Given  $\epsilon > 0$  there exists  $N_1$  such that  $|x_n - x_m| \leq \epsilon/2$  for all  $n, m$  with  $n, m \geq N_1$ . Likewise, there exists  $N_2$  such that  $|y_n - y_m| \leq \epsilon/2$  for all  $n, m$  with  $n, m \geq N_2$ . Thus, if  $M = \max(N_1, N_2)$  and  $n, m \geq M$  we have

$$|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus,  $\overline{x} + \overline{y}$  is a Cauchy sequence. Turning to the product, we note that the sequences  $\{|x_n|\}$  and  $\overline{y} = \{|y_n|\}$ , being Cauchy (as the method of Exercise 2 shows), are bounded. Let  $U > 0$  be a common upper bound. Given  $\epsilon > 0$  there exists an  $N_1$  such that  $|x_n - x_m| \leq \epsilon/(2U)$  for all  $n, m \geq N_1$  and an  $N_2$  such that  $|y_n - y_m| \leq \epsilon/(2U)$  for all  $n, m \geq N_2$ . Thus, for  $n, m \geq \max(N_1, N_2)$  we have

$$\begin{aligned} |x_n y_n - x_m y_m| &\leq |(x_n y_n - x_n y_m) + (x_n y_m - x_m y_m)| \\ &= |x_n| |y_n - y_m| + |y_m| |x_n - x_m| \\ &\leq U \cdot \frac{\epsilon}{2U} + U \cdot \frac{\epsilon}{2U} \\ &= \epsilon. \end{aligned}$$

5. Let  $\overline{X}$  denote the set of Cauchy sequences with values in an ordered field. Let  $P$  be the subset of  $\overline{X}$  comprising all  $\overline{x} = \{x_n\} \in \overline{X}$  with the property that there exists an  $\epsilon > 0$  and an  $N$  such that  $x_n \geq \epsilon$  for all  $n \geq N$ . Suppose that  $\overline{x} = \{x_n\} \in P$  and  $\overline{y} = \{y_n\} \in P$ . Suppose that  $\overline{z} = \{z_n\}$  is equivalent to  $\overline{x}$  in the sense of Exercise 3. Show that  $\overline{x} + \overline{y}$ ,  $\overline{x} \cdot \overline{y}$ , and  $\overline{z}$  belong to  $P$ .

**Solution** There exists an  $\epsilon_x > 0$  and an  $N_x$  such that  $x_n \geq \epsilon_x$  for all  $n \geq N_x$ . Also, there exists an  $\epsilon_y > 0$  and an  $N_y$  such that  $y_n \geq \epsilon_y$  for all  $n \geq N_y$ . As a result,  $x_n + y_n \geq \epsilon_x + \epsilon_y$  for all  $n \geq \max(N_x, N_y)$ . It follows that  $\{x_n + y_n\} \in P$ . Additionally,  $x_n y_n \geq \epsilon_x \epsilon_y$  for all  $n \geq \max(N_x, N_y)$ . It follows that  $\{x_n y_n\} \in P$ . Finally, if  $\{z_n\} \sim \{x_n\}$ , there exists an  $N_3$  such that  $|z_n - x_n| \leq \epsilon_x/2$  for all  $n \geq N_3$ . Therefore, for  $n \geq \max(N_1, N_3)$  we have

$$z_n = x_n + (z_n - x_n) \geq x_n - |z_n - x_n| \geq \epsilon_x - \frac{1}{2}\epsilon_x = \frac{1}{2}\epsilon_x,$$

which shows that  $\{z_n\} \in P$ .

6. Let  $s_0 = d_0 = 1$ . For  $n \geq 1$  let  $s_n = s_{n-1} + d_{n-1}$ , and  $d_n = 2s_{n-1} + d_{n-1}$ . Use mathematical induction to show that  $d_n^2 = 2s_n^2 + (-1)^{n+1}$ . Prove that the sequence  $\{d_n/s_n\}$  of rational numbers is a Cauchy sequence. Prove that the sequence  $\{d_n^2/s_n^2\}$  of rational numbers converges to 2. (This last result shows that the Cauchy sequence  $\{d_n/s_n\}$  does not converge in the rationals. The rational numbers are therefore not complete.)

**Solution** For  $n = 0$ , the equation  $d_n^2 = 2s_n^2 + (-1)^{n+1}$  becomes  $d_0^2 = 2s_0^2 - 1$ , which is true because  $s_0 = d_0 = 1$ . The asserted equation is more interesting for  $n = 1$ . Indeed, for  $n = 1$ , we have  $d_1^2 = (2s_0 + d_0)^2 = 3^2 = 2 \cdot 2^2 + 1 = 2 \cdot (s_0 + d_0)^2 + 1 = 2s_1^2 + (-1)^{1+1}$ . Next, we turn to the induction step. Suppose that  $d_N^2 = 2s_N^2 + (-1)^{N+1}$  for some value  $N$  of  $n$ . Squaring each side of the recurrence relations for  $d_{N+1}$  and  $s_{N+1}$  we obtain

$$d_{N+1}^2 = (2s_N + d_N)^2 = 4s_N^2 + 4s_N d_N + d_N^2$$

and

$$2s_{N+1}^2 = 2(s_N + d_N)^2 = 2s_N^2 + 4s_N d_N + 2d_N^2.$$

Subtracting the second of these equations from the first, we see that

$$d_{N+1}^2 - 2s_{N+1}^2 = 2s_N^2 - d_N^2.$$

Thus,  $d_{N+1}^2 - 2s_{N+1}^2 = -(-1)^{N+1} = (-1)^{N+2}$ , which is the asserted equation for  $n = N + 1$ . If we divide both sides of the equation  $d_n^2 = 2s_n^2 + (-1)^{n+1}$  by  $s_n^2$ , we obtain  $d_n^2/s_n^2 = 2 \pm 1/s_n^2$ . The equation  $s_n = s_{n-1} + d_{n-1}$  and the initializations  $s_0 = d_0 = 1$  lead to the estimate  $s_n \geq n + 1$  for all  $n \in \mathbb{N}$ . It follows that  $\lim_{n \rightarrow \infty} d_n^2/s_n^2 = 2$ . To see that  $\{d_n/s_n\}$  is a Cauchy sequence, observe that

$$\left(\frac{d_n}{s_n} - \frac{d_m}{s_m}\right) \left(\frac{d_n}{s_n} + \frac{d_m}{s_m}\right) = \frac{d_n^2}{s_n^2} - \frac{d_m^2}{s_m^2} = 2 + \frac{(-1)^{n+1}}{s_n^2} - \left(2 + \frac{(-1)^{m+1}}{s_m^2}\right) = \frac{\pm 1}{s_n^2} + \frac{\pm 1}{s_m^2}.$$

Since  $\lim_{n \rightarrow \infty} d_n^2/s_n^2 = 2$  there exists an  $N_1$  such that  $d_n/s_n > 1$  for  $n \geq N_1$ . Also, given  $\epsilon > 0$ , there exists an  $N_2$  such that  $1/s_n^2 \leq \epsilon$ . For  $n \geq \max(N_1, N_2)$  we have

$$\left|\frac{d_n}{s_n} - \frac{d_m}{s_m}\right| \leq \frac{1}{(d_n/s_n + d_m/s_m)} \left(\frac{1}{s_n^2} + \frac{1}{s_m^2}\right) \leq \frac{1}{2}(\epsilon + \epsilon) = \epsilon.$$

It follows that  $\{d_n/s_n\}$  is a Cauchy sequence.