Math 4111 Fall 2008
Exercises September 9

1. The Well-ordering Principle of the natural numbers $\mathbb{N}$ states that every nonempty subset of $\mathbb{N}$ has a least element. Use this principle together with the completeness property of the real number system to show that for every element $y \in \mathbb{R}$ there is an $m \in \mathbb{Z}$ such that $m \leq y < m+1$. Deduce that for every $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ there is an $m \in \mathbb{Z}$ such that $|x - m/n| < 1/n$. Show that for every $x \in \mathbb{R}$ and $\epsilon > 0$ there is an $a \in \mathbb{Q}$ such that $|x - a| < \epsilon$. (This proves that $\mathbb{Q}$ is dense in $\mathbb{R}$).

Solution If $y \in \mathbb{Z}$ then $y \leq y < y+1$. We may therefore assume that $y \in \mathbb{R} \setminus \mathbb{Z}$. We are therefore to show that there is an $m \in \mathbb{Z}$ such that $m < y < m+1$. Since, $m < y < m+1$ if and only if $-(m+1) < -y < -m$, we see that the assertion for $y < 0$ holds if the assertion for $y > 0$ holds. Assume then that $y \in \mathbb{R}^+ \setminus \mathbb{Z}$. The completeness property of $\mathbb{R}$ implies that $\mathbb{R}$ is archimedian. Thus, the set $S_y = \{k \in \mathbb{Z}^+ : y < k\}$ is nonempty. Let μ be the least element of S_y. Then $\mu - 1 \notin S_y$ so $\mu - 1 < y$. Letting $m = \mu - 1$, we have $m < y < m+1$, as required. Next, let $n \in \mathbb{Z}^+$. By the first part of this exercise applied to $y = nx$, there exists $m \in \mathbb{Z}$ such that $m \leq nx < m+1$. Thus, $0 \leq nx - m < 1$, and $0 \leq x - m/n < 1/n$, as required. Finally, $x \in \mathbb{R}$ and $\epsilon > 0$, then, by the archimedean property, there is an $n \in \mathbb{Z}^+$ such that $1/\epsilon < n$. It follows that $1/n < \epsilon$. By the preceding part of this exercise, we may find $a = m/n \in \mathbb{Q}$ such that $0 \leq x - a < 1/n < \epsilon$.

2. Suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space (X, d). Show that $\{d(x_n, y_n)\}$ is a convergent sequence in $\mathbb{R}$.

Solution Using the triangle inequality twice we estimate

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_n) \leq d(x_n, x_m) + (d(x_m, y_m) + d(y_m, y_n)).$$

Therefore,

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n).$$

By interchanging m and n in this inequality and noting that there is no change on the right side, we obtain

$$-(d(x_n, y_n) - d(x_m, y_m)) \leq d(x_n, x_m) + d(y_m, y_n).$$

Together these inequalities give us

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n).$$

Given $\epsilon > 0$ there is an integer N_x such that $d(x_n, x_m) \leq \epsilon/2$ for all $n, m \geq N_x$ and an integer N_y such that $d(y_m, y_n) \leq \epsilon/2$ for all $n, m \geq N_y$. Then, for all $n, m \geq \max(N_x, N_y)$ we have

$$|d(x_n, y_n) - d(x_m, y_m)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

3. Let $s_1 = \sqrt{2}$. For $n \geq 2$ let $s_n = \sqrt{2 + s_{n-1}}$. Show that $\{s_n\}$ converges in $\mathbb{R}$.

Solution We first show by mathematical induction that $\{s_n\}$ is bounded by 2. The basis step $s_1 \leq 2$ is plain. For the induction step suppose that $s_N \leq 2$ for some specific value N of n. Then $s_{N+1} = \sqrt{2 + s_N} \leq \sqrt{2+2} = 2$, which establishes the induction step and completes the proof that $s_n \leq 2$ for all values of n. Next we use the bound we have just obtained to prove that $\{s_n\}$ is an increasing sequence. We have

$$s_n = \frac{1}{s_n} \cdot s_n^2 = \frac{1}{s_n}(2 + s_{n-1}) \geq \frac{1}{s_n}(s_{n-1} + s_{n-1}) = \frac{1}{s_n}(2s_{n-1}) \geq \frac{1}{s_n}(s_n s_{n-1}) = s_{n-1}.$$

Since $\{s_n\}$ is monotone increasing and $\mathbb{R}$ is complete, we conclude that $\{s_n\}$ converges in $\mathbb{R}$.

4. Suppose that A and B are bounded subsets of $\mathbb{R}$. Let $A+B = \{a+b : a \in A, b \in B\}$. Show that $\sup(A+B) = \sup(A) + \sup(B)$.

Solution If $a \in A$ and $b \in B$ then $a \leq \sup(A)$ and $b \leq \sup(B)$. It follows that $a + b \leq \sup(A) + \sup(B)$ for all $a \in A, b \in B$. This means that $\sup(A) + \sup(B)$ is an upper bound for $A + B$. Therefore $\sup(A + B) \leq \sup(A) + \sup(B)$. To obtain equality, we demonstrate the reverse inequality. Let $\epsilon > 0$. Then there exist a_1 and b_1 such that $a_1 + \epsilon/2 > \sup(A)$ and $b_1 + \epsilon/2 > \sup(B)$. Thus, $(a_1 + \epsilon/2) + (b_1 + \epsilon/2) > \sup(A) + \sup(B)$. In other words, $a_1 + b_1 > \sup(A) + \sup(B) - \epsilon$. It follows that $\sup(A + B) \geq \sup(A) + \sup(B) - \epsilon$ for every $\epsilon > 0$. We conclude that $\sup(A + B) \geq \sup(A) + \sup(B)$, which completes the proof of the asserted equality.

5. If ϕ is a function with domain S, let $\phi(S)$ denote the image of ϕ. (In general, if $T \subset S$, then $\phi(T) = \{\phi(t) : t \in T\}$). If f and g are two real-valued functions with domain S, let $f + g$ be the real-valued function with domain S that is defined by $(f + g)(s) = f(s) + g(s) \quad (s \in S)$. Suppose that $f(S)$ and $g(S)$ are nonempty bounded sets. Show that

$$\inf(f(S)) + \inf(g(S)) \leq \inf((f+g)(S)) \leq \inf(f(S)) + \sup(g(S)) \leq \sup((f+g)(S)) \leq \sup(f(S)) + \sup(g(S)).$$

Solution Before proceding to the asserted inequalities, we will prove a lemma that will be useful here. To wit, if $A \subset \mathbb{R}$, if $c \in \mathbb{R}$, and if $A + c = \{x \in \mathbb{R} \,|\, x = a + c \text{ for some } a \in A\}$, then $\inf(A) + c = \inf(A + c)$. As we often do to prove such an equality, we prove that each side is no less than the other. If $a \in A$, then $\inf(A) + c \leq a + c$. Therefore $\inf(A) + c$ is a lower bound for the set $A + c$. It follows that $\inf(A) + c \leq \inf(A + c)$. To derive the reverse inequality $\inf(A + c) \leq \inf(A) + c$, let $\epsilon > 0$ and choose $a_1 \in A$ such that $a_1 < \inf(A) + \epsilon$. Then $a_1 + c \leq \inf(A) + c + \epsilon$. This tells us that $\inf(A) + c + \epsilon$ is not a lower bound for the set $A + c$. Therefore $\inf(A + c) \leq \inf(A) + c + \epsilon$. Since this inequality is true for every $\epsilon > 0$, we conclude that $\inf(A + c) \leq \inf(A) + c$. The two inequalities that we have obtained, $\inf(A) + c \leq \inf(A + c)$ and $\inf(A + c) \leq \inf(A) + c$, show that $\inf(A) + c = \inf(A + c)$. Now we proceed to the leftmost asserted inequality of the exercise, reserving the result of the lemma for the next inequality. If $s \in S$, then $\inf(f(S)) \leq f(s)$ and $\inf(g(S)) \leq g(s)$. Therefore,

$$\inf(f(S)) + \inf(g(S)) \leq f(g) + g(s) = (f+g)(s) \qquad \text{for all } s \in S.$$

In other words, $\inf(f(S)) + \inf(g(S))$ is a lower bound for the set $(f + g)(S)$. It follows that $\inf(f(S)) + \inf(g(S)) \leq \inf((f + g)(S))$. Next, suppose that $s \in S$. We have

$$\inf((f+g)(S)) \leq (f+g)(s) = f(s) + g(s) \leq f(s) + \sup(g(S)) \qquad \text{for all } s \in S.$$

Thus, $\inf((f + g)(S))$ is a lower bound for the set $A + c$ where $A = f(S)$ and $c = \sup(g(S))$. Thus, $\inf((f + g)(S)) \leq \inf(A + c)$. By the lemma that we proved at the start of this exercise, $\inf(A + c) = \inf(A) + c$. Putting together this equation with the previous inequality, we obtain $\inf((f + g)(S)) \leq \inf(A) + c = \inf(f(S)) + \sup(g(S))$, which is the second of the asserted inequalities. We now apply this inequality to the functions $-g$ and $-f$. It states: $\inf((-g + (-f))(S)) \leq \inf(-g(S)) + \sup(-f(S))$, or $\inf(-(g + f)(S)) \leq \inf(-g(S)) + \sup(-f(S))$, or $-\sup((g + f)(S)) \leq -\sup(g(S)) - \inf(f(S))$, or $\inf(f(S)) + \sup(g(S)) \leq \sup((f + g)(S))$, which is the third of the required inequalities. We use the same technique to obtain the last required inequality. That is, we apply the first inequality of the functions $-f$ and $-g$. Thus $\inf(-f(S)) + \inf(-g(S)) \leq \inf(-(f + g)(S))$, or $-\sup(f(S)) - \sup(g(S)) \leq -\sup((f + g)(S))$, or $\sup((f + g)(S)) \leq \sup(f(S)) + \sup(g(S))$.