Math 4111 Fall 2008
Exercises September 16

1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Let $M = \limsup_{n\to\infty} x_n$. Show that $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = M$.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Let $M = \limsup_{n\to\infty} x_n$. Suppose that $\epsilon > 0$. Show that $x_n \leq M + \epsilon$ for all but finitely many n and $x_n \geq M - \epsilon$ for infinitely many n.

3. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in $\mathbb{R}$. Let $\xi = \limsup_{n\to\infty} x_n$ and $\eta = \limsup_{n\to\infty} x_n$. Show that $\limsup_{n\to\infty} (x_n + y_n) \leq \xi + \eta$.

4. In the preceding exercise show that inequality may occur. Show that equality does occur if at least one of the sequences is convergent.

5. Define $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ by $x_1 = 0$, $x_{2n} = (1/2)\, x_{2n-1}$ for $n \in \mathbb{Z}^+$, and $x_{2n+1} = 1/2 + x_{2n}$ for $n \in \mathbb{Z}^+$. Find $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$.