Math 4111 Fall 2008
Exercises September 16

1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Let $U = \limsup_{n\to\infty} x_n$. Show that $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = U$.

Solution Let $u_N = \sup\left(\{x_n\}_{n=N}^{\infty}\right)$. Then $\lim_{N\to\infty} u_N = U$. Thus, there exists a N_1 such that $0 \le u_{N_1} - U \le 1/2$. Also, there exists an N_1' with $N_1' \ge N_1$ and $0 \le u_{N_1} - x_{N_1'} \le 1/2$. Let $n_1 = N_1'$. It follows that

$$|x_{n_1} - U| \le (u_{N_1} - U) + \left(u_{N_1} - x_{N_1'}\right) \le 1.$$

Next, there exists N_2 such that $N_2 > n_1$ and $0 \le u_{N_2} - U \le 1/4$. Also, there exists an N_2' with $N_2' \ge N_2$ and $0 \le u_{N_2} - x_{N_2'} \le 1/4$. Let $n_2 = N_2'$. We have

$$|x_{n_2} - U| \le (u_{N_2} - U) + \left(u_{N_2} - x_{N_2'}\right) \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Next, there exists N_3 such that $N_3 > n_2$ and $0 \le u_{N_3} - U \le 1/6$. Also, there exists an N_3' with $N_3' \ge N_3$ and $0 \le u_{N_3} - x_{N_3'} \le 1/6$. Let $n_3 = N_3'$. We have

$$|x_{n_3} - U| \le (u_{N_3} - U) + \left(u_{N_3} - x_{N_3'}\right) \le \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Continuing in this manner, we obtain a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $|x_{n_k} - U| \le 1/k$. Clearly, $\lim_{k\to\infty} x_{n_k} = U$.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Let $U = \limsup_{n\to\infty} x_n$. Suppose that $\epsilon > 0$. Show that $x_n \le U + \epsilon$ for all but finitely many n and $x_n \ge U - \epsilon$ for infinitely many n.

Solution Suppose that $\epsilon > 0$ and $U = \limsup_{n\to\infty} x_n$. If $x_n > U + \epsilon$ for infinitely many n, then for every N there is an n with $n \ge N$ and $x_n > U + \epsilon$. Thus $U + \epsilon$ is not an upper bound for $\{x_n\}_{n=N}^{\infty}$ for any N. It follows that $u_N = \sup\left(\{x_n\}_{n=N}^{\infty}\right) \ge U + \epsilon$. Therefore, $U = \lim_{N\to\infty} u_N \ge U + \epsilon > U$, which is a contradiction. This proves the first assertion. The second assertion is also proved by contradiction. If $x_n \ge U - \epsilon$ for only finitely many n, then there is a largest value N of n such that $x_N \ge U - \epsilon$. As a result, if $M > N$, then $x_n < U - \epsilon$ for all $n \ge M$. It follows that $u_M = \sup\left(\{x_n\}_{n=M}^{\infty}\right) \le U - \epsilon$ for all M with $M > N$. Therefore, $U = \lim_{N\to\infty} u_N \le U - \epsilon < U$, which is a contradiction. This proves the second assertion.

3. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in $\mathbb{R}$. Let $\xi = \limsup_{n\to\infty} x_n$ and $\eta = \limsup_{n\to\infty} y_n$. Show that $\limsup_{n\to\infty}(x_n + y_n) \le \xi + \eta$.

Solution Let $\epsilon > 0$. By the preceding exercise, there exists an N such that $x_n \le \xi + \epsilon/2$ and $y_n \le \eta + \epsilon/2$ for all $n \ge N$. Therefore, $x_n + y_n \le \xi + \eta + \epsilon$ for all $n \ge N$. Thus, $\sup\left(\{x_n + y_n\}_{n=M}^{\infty}\right) \le \xi + \eta + \epsilon$ for all $M \ge N$. It follows that

$$\limsup_{n\to\infty}(x_n + y_n) = \lim_{M\to\infty} \sup\left(\{x_n + y_n\}_{n=M}^{\infty}\right) \le \xi + \eta + \epsilon.$$

Since ϵ is an arbitrary positive number we deduce that $\limsup_{n\to\infty}(x_n + y_n) = \lim_{M\to\infty} \sup\left(\{x_n + y_n\}_{n=M}^{\infty}\right) \le \xi + \eta$.

4. In the preceding exercise show that inequality may occur. Show that equality does occur if at least one of the sequences is convergent.

Solution Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then $\xi = \limsup_{n\to\infty} x_n = 1$ and $\eta = \limsup_{n\to\infty} y_n = 1$. Since $x_n + y_n = 0$ for all n we have $\limsup_{n\to\infty}(x_n + y_n) = 0 < \xi + \eta$. Now suppose that $\{y_n\}$ is convergent. Then every subsequence of $\{y_n\}$ converges to $\lim_{n\to\infty} y_n$ Since there is a subsequence that converges to $\eta = \limsup_{n\to\infty} y_n$, we conclude that $\eta = \lim_{n\to\infty} y_n$. Let $\zeta = \limsup_{n\to\infty}(x_n + y_n)$. We know from the first exercise that there is a subsequence $\{x_{n_k}\}$ such that $\xi = \lim_{k\to\infty} x_{n_k}$. Therefore $\lim_{k\to\infty}(x_{n_k} + y_{n_k}) = \lim_{k\to\infty} x_{n_k} + \lim_{k\to\infty} y_{n_k} = \xi + \eta$. Because every subsequential limit of $\{x_n + y_n\}$ is no greater than ζ, we have $\xi + \eta \le \zeta$. We also know from the preceding exercise that $\zeta \le \xi + \eta$. From the inequalities $\xi + \eta \le \zeta \le \xi + \eta$ it follows that $\zeta = \xi + \eta$.

5. Define $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ by $x_1 = 0$, $x_{2n} = (1/2)\, x_{2n-1}$ for $n \in \mathbb{Z}^+$, and $x_{2n+1} = 1/2 + x_{2n}$ for $n \in \mathbb{Z}^+$. Find $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$.

Solution The first few terms are

1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x_n	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{15}{16}$	$\frac{15}{32}$	$\frac{31}{32}$	$\frac{31}{64}$	$\frac{63}{64}$	$\frac{63}{128}$

From the evidence it appears that $x_{2n} \uparrow 1/2$ (increases to a limit of $1/2$) and $x_{2n-1} \uparrow 1$. Let us prove this. Let $y_k = x_{2k}$. Then

$$y_{k+1} = x_{2k+2} = \frac{1}{2} x_{2k+1} = \frac{1}{2}\left(\frac{1}{2} + x_{2k}\right) = \frac{1}{4} + \frac{1}{2} x_{2k} = \frac{1}{4} + \frac{1}{2} y_k.$$

Thus, if $y_K < 1/2$ for some specific value K of k, we have $y_{K+1} < 1/4 + (1/2)(1/2) = 1/2$. Since $y_1 = x_2 = 0 < 1/2$, it follows by mathematical induction that $y_k < 1/2$ for all $k \in \mathbb{Z}^+$. From this bound, we see that

$$y_{k+1} - y_k = \left(\frac{1}{4} + \frac{1}{2} y_k\right) - y_k = \frac{1}{4} - \frac{1}{2} y_k > \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} = 0.$$

Therefore $\{y_k\}$ is an increasing sequence. It follows that there is an ℓ such that $y_k \to \ell$.To calculate ℓ, note that

$$\ell = \lim_{k\to\infty} y_k = \lim_{k\to\infty} y_{k+1} = \lim_{k\to\infty} \left(\frac{1}{4} + \frac{1}{2} y_k\right) = \frac{1}{4} + \frac{1}{2} \lim_{k\to\infty} y_k = \frac{1}{4} + \frac{1}{2}\ell.$$

This equation gives us $\ell = 1/2$. Let $z_k = x_{2k-1}$. Then

$$z_{k+1} = x_{2k+1} = \frac{1}{2} + x_{2k} = \frac{1}{2} + \frac{1}{2} x_{2k-1} = \frac{1}{2} + \frac{1}{2} z_k.$$

Thus, if $z_K < 1$ for some specific value K of k, we have $z_{K+1} < 1/2 + (1/2) \cdot 1 = 1$. Since $z_1 = x_1 = 0 < 1$, it follows by mathematical induction that $z_k < 1$ for all $k \in \mathbb{Z}^+$. From this bound, we see that

$$z_{k+1} - z_k = \left(\frac{1}{2} + \frac{1}{2} z_k\right) - z_k = \frac{1}{2} - \frac{1}{2} z_k > \frac{1}{2} - \frac{1}{2} \cdot 1 = 0.$$

Therefore $\{z_k\}$ is an increasing sequence. It follows that there is a u such that $y_k \to u$.To calculate u, note that

$$u = \lim_{k\to\infty} z_k = \lim_{k\to\infty} z_{k+1} = \lim_{k\to\infty} \left(\frac{1}{2} + \frac{1}{2} z_k\right) = \frac{1}{2} + \frac{1}{2} \lim_{k\to\infty} z_k = \frac{1}{2} + \frac{1}{2} u.$$

This equation gives us $u = 1$. By accounting for both the even and odd indices of $\{x_n\}$, we have accounted for all indices of $\{x_n\}$ and can be sure that there are only two subsequential limits: $1/2$ and 1. The smaller, $1/2$, is the smallest subsequential limit and the larger is the largest subsequential limit. Therefore $\liminf_{n\to\infty} x_n = 1/2$ and $\limsup_{n\to\infty} x_n = 1$.