Math 4111 Fall 2008
Exercises September 25

1. Let S be a subset of a metric space (X, d). Show that $\partial(S) = \overline{S} \cap \overline{\complement S}$.

Solution By definition, a point x belongs to $\partial(S)$ if for every $r > 0$ the two intersections $B_r(x) \cap S$ and $B_r(x) \cap \complement S$ are nonempty. Since S and $\complement S$ are interchangable in this definition, we deduce that $\partial(S) = \partial(\complement S)$. By definition $\partial(\complement S) \subset \overline{\complement S}$. Since $\partial(S) = \partial(\complement S)$ we have $\partial(S) \subset \overline{\complement S}$. By definition it is also true that $\partial(S) \subset \overline{S}$. Combining the two inclusions $\partial(S) \subset \overline{S}$ and $\partial(S) \subset \overline{\complement S}$, we conclude that $\partial(S) \subset \overline{S} \cap \overline{\complement S}$. This is one half of the set equality we require. To complete the proof, we must show the reverse containment: $\overline{S} \cap \overline{\complement S} \subset \partial(S)$. Because $\overline{S} = S \cup \partial(S)$ and $\overline{\complement S} = \complement S \cup \partial(\complement S) = \complement S \cup \partial(S)$, we are to prove that $(S \cup \partial(S)) \cap (\complement S \cup \partial(S)) \subset \partial(S)$. This amounts to $\left(S \cap (\complement S \cup \partial(S))\right) \cup \left(\partial(S) \cap (\complement S \cup \partial(S))\right) \subset \partial(S)$. In other words, we must prove the two set containments, $S \cap (\complement S \cup \partial(S)) \subset \partial(S)$ and $\partial(S) \cap (\complement S \cup \partial(S)) \subset \partial(S)$. The second of these containments is clear, since $\partial(S) \cap Z \subset \partial(S)$ whatever set Z may be. As for the first set containment, notice that $S \cap \complement S = \emptyset$. Therefore $S \cap (\complement S \cup \partial(S)) = S \cap \partial(S)$, which clearly is a subset of $\partial(S)$.

2. Prove the following two formulas for the interior S° of any set S:

$$S^{\circ} = S \setminus \partial(S) = \complement\left(\overline{\complement S}\right).$$

Solution Suppose that $x \in S^{\circ}$. Then $x \in S$ because $S^{\circ} \subset S$. By definition of S°, there exists an $r > 0$ with $B_r(x) \subset S$. Therefore $x \notin \partial(S)$. In summary, $x \in S$ and $x \notin \partial(S)$. This means that $x \in S \setminus \partial(S)$. Since x is an arbitrary point of S°, we conclude that $S^{\circ} \subset S \setminus \partial(S)$. This containment gives us one half of the required set equality $S^{\circ} = S \setminus \partial(S)$. To complete the proof of the equality, we must demonstrate the reverse containment: $S \setminus \partial(S) \subset S^{\circ}$. Toward that end, let $x \in S \setminus \partial(S)$. Since $x \notin \partial(S)$ there is an $r > 0$ such that either $B_r(x) \cap S = \emptyset$ or $B_r(x) \cap \complement S = \emptyset$. Since $x \in S$ we see that $\{x\} \subset B_r(x) \cap S$ and therefore $B_r(x) \cap S \neq \emptyset$. It follows that $B_r(x) \cap \complement S = \emptyset$. This means that $B_r(x) \subset \complement(\complement S) = S$. This containment tells us that $x \in S^{\circ}$. Since an arbitrary point x of $S \setminus \partial(S)$ must belong to S° as well, we conclude that $S \setminus \partial(S) \subset S^{\circ}$. Together, the two proven containments $S^{\circ} \subset S \setminus \partial(S)$ and $S \setminus \partial(S) \subset S^{\circ}$ tell us that $S^{\circ} = S \setminus \partial(S)$. Next, observe that

$$\complement\left(\overline{\complement S}\right) = \complement\left(\complement S \cup \partial(\complement S)\right) = \complement\left(\complement S \cup \partial(S)\right) = \left(\complement(\complement S)\right) \cap \complement\partial(S) = S \cap \complement\partial(S) = S \setminus \partial(S),$$

so the second set equality, $S^{\circ} = \complement\left(\overline{\complement S}\right)$, follows from the first.

3. Prove that $\complement(\partial(S)) = S^{\circ} \cup (\complement S)^{\circ}$ for any subset S of $\mathbb{R}^N$.

Solution If $x \in \complement(\partial(S))$, that is, if $x \notin \partial(S)$, then there is an $r > 0$ such that $B_r(x) \cap S = \emptyset$ or $B_r(x) \cap \complement S = \emptyset$. In other words, $B_r(x) \subset S$ or $B_r(x) \subset \complement S$. It follows that $x \in S^{\circ}$ or $x \in (\complement S)^{\circ}$. Thus, $\complement(\partial(S)) \subset S^{\circ} \cup (\complement S)^{\circ}$. In the other direction, if $x \in S^{\circ}$ or, respectively, $x \in (\complement S)^{\circ}$, then there is an $r > 0$ such that $B_r(x) \subset S$ or, respectively, $B_r(x) \subset \complement S$. In the first situation $B_r(x) \cap \complement S = \emptyset$. In the second situation $B_r(x) \cap S = \emptyset$. Either case implies $x \notin \partial(S)$, which is to say that $x \in \complement(\partial(S))$. Thus $S^{\circ} \cup (\complement S)^{\circ} \subset \complement(\partial(S))$, completing the proof that $\complement(\partial(S)) = S^{\circ} \cup (\complement S)^{\circ}$.

4. Prove that a vector subspace of an inner product space is closed.

Solution Suppose that $\mathcal{U}$ is a vector subspace of an inner product space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$. Suppose that $\mathbf{x} \in \mathcal{V}$ and $\{\mathbf{x}_n\} \subset \mathcal{U}$ with $\mathbf{x}_n \to \mathbf{x}$. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be an orthonormal basis for $\mathcal{U}$ and extend this to an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_d$ of $\mathcal{V}$. For each fixed $j = m+1, \ldots, d$, we have $\langle \mathbf{x}_n, \mathbf{u}_j \rangle \to \langle \mathbf{x}, \mathbf{u}_j \rangle$ as $n \to \infty$. But $\langle \mathbf{x}_n, \mathbf{u}_j \rangle = 0$ for all n since $\{\mathbf{x}_n\} \subset \mathcal{U}$. Therefore $\langle \mathbf{x}, \mathbf{u}_j \rangle = 0$ for $j = m+1, \ldots, d$. Thus,

$$\mathbf{x} = \sum_{j=1}^{d} \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j = \sum_{j=1}^{m} \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j + \sum_{j=m+1}^{d} \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j = \sum_{j=1}^{m} \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j \in \mathcal{U}.$$

Having shown that $\mathcal{U}$ contains all its limit points, we conclude that $\mathcal{U}$ is closed.

5. In a metric space (X, d), the distance $d(p, A)$ of a point $p \in X$ to a nonempty subset A of X is defined by $d(p, A) = \inf(\{d(x, a) \mid a \in A\})$. Prove that $\overline{A} = \{x \in X \mid d(x, A) = 0\}$.

Solution If $x \in A$ then $d(x, a) = 0$ for $a = x \in A$, and so $d(x, A) = 0$. If $x \in \partial(A)$, then $B_{1/n}(x) \cap A \neq \emptyset$ for every $n \in \mathbb{Z}^+$. Thus for every $n \in \mathbb{Z}^+$ there exists an $a_n \in A$ with $d(x, a_n) < 1/n$. Thus $\inf(\{d(x, a) \mid a \in A\}) < 1/n$ for all $n \in \mathbb{Z}^+$. This implies $d(p, A) = \inf(\{d(x, a) \mid a \in A\}) = 0$. Since $\overline{A} = A \cup \partial(A)$, we have so far proved that $\overline{A} \subset \{x \in X \mid d(x, A) = 0\}$. To complete the proof of the equality of these two sets we must prove the reverse set containment, $\{x \in X \mid d(x, A) = 0\} \subset \overline{A}$. Toward that end, suppose that $x \in \{x \in X \mid d(x, A) = 0\} \cap \complement A$. Let $r > 0$. Then $B_r(x) \cap \complement A \neq \emptyset$ since x is an element in the intersection. Since $\inf(\{d(x, a) \mid a \in A\}) = 0$ there exists $a \in A$ with $d(x, a) < r$. Therefore $B_r(x) \cap A \neq \emptyset$ since a is an element in the intersection. In summary, we have shown that for every $r > 0$ both $B_r(x) \cap A \neq \emptyset$ and $B_r(x) \cap \complement A \neq \emptyset$. In other words, $x \in \partial(A)$. The upshot of this argument is that if $x \in x \in \{x \in X \mid d(x, A) = 0\}$, then $x \in A$ or $x \in \partial(A)$. This means that $\{x \in X \mid d(x, A) = 0\} \subset \overline{A}$, completing the proof that $\overline{A} = \{x \in X \mid d(x, A) = 0\}$.