Math 4111 Fall 2008
Exercises October 9

1. Suppose that X is a metric space and that $E \subset X$. A point $p \in X$ is a *condensation point* of E if $B_r(p) \cap E$ is uncountable for every $r > 0$. Prove that the set of condensation points of E is closed.

2. Prove that the set of condensation points of a set $E \subset \mathbb{R}^d$ is nonempty if E is uncountable.

3. Suppose that X is a compact metric space and that $F \subset X$ is closed. Prove that F is compact.

4. Suppose that X is a metric space, that $S \subset X$, and that $K \subset S$ is relatively compact in S. Prove that K is compact.

5. Suppose that X and Y are sets, that $f : X \to Y$, that $\mathcal{P}(X)$ is the power set of X, that $\mathcal{P}(Y)$ is the power set of Y, that $A \subset X$, that $B \subset Y$, that $\{A_\alpha : \alpha \in \mathcal{A}\} \subset \mathcal{P}(X)$, and that $\{B_\beta : \beta \in \mathcal{B}\} \subset \mathcal{P}(Y)$. Prove the following relations for the in induced functions $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$:

$$f\left(\bigcup_{\alpha\in\mathcal{A}} A_a\right) = \bigcup_{\alpha\in\mathcal{A}} f(A_a),$$

$$f\left(\bigcap_{\alpha\in\mathcal{A}} A_a\right) \subset \bigcap_{\alpha\in\mathcal{A}} f(A_a),$$

$$f\left(\bigcap_{\alpha\in\mathcal{A}} A_a\right) = \bigcap_{\alpha\in\mathcal{A}} f(A_a) \text{ if } f \text{ is injective},$$

$$f^{-1}\left(\bigcup_{\beta\in\mathcal{B}} B_\beta\right) = \bigcup_{\beta\in\mathcal{B}} f^{-1}(B_\beta),$$

$$f^{-1}\left(\bigcap_{\beta\in\mathcal{B}} B_\beta\right) = \bigcap_{\beta\in\mathcal{B}} f^{-1}(B_\beta),$$

$$f\left(f^{-1}(B)\right) \subset B,$$

$$f\left(f^{-1}(B)\right) = B \text{ if } f \text{ is surjective},$$

$$A \subset f^{-1}(f(A)), \text{ and}$$

$$A = f^{-1}(f(A)) \text{ if } f \text{ is injective}.$$