1. Consider the following overlapping generations growth model. Time is indexed by $t = 0, 1, 2, \ldots$, and in period $t$ there are $L_t$ two-period-lived consumers born, where $L_t = (1 + n)^t L_0$, with $L_0$ given and $n > 0$. At $t = 0$ there are some one-period-lived old agents who are collectively endowed with $K_0$ units of capital, and who maximize consumption at $t = 0$. A consumer born in period $t$ has preferences given by

$$u(c^y_t, c^o_{t+1}) = 2(c^y_t)^{\frac{1}{2}} + 2(c^o_{t+1})^{\frac{1}{2}},$$

where $c^y_t$ ($c^o_t$) denotes consumption of a young (old) consumer in period $t$. Each consumer has one unit of time available when young. The production technology is given by

$$Y_t = K_t^\alpha L_t^{1-\alpha},$$

where $Y_t$ is output, $K_t$ is the capital input, and $L_t$ is the labor input. One unit of the consumption good can be converted into one unit of capital, and vice-versa. Let $\tau^y_t$ ($\tau^o_t$) denote the lump-sum tax paid by a young (old) agent in period $t$, and let $w_t$ denote the wage rate and $r_t$ the rental rate on capital.

(a) Determine the capital/labor ratio, consumption of the young, and consumption of the old, in a socially optimal steady state.

(b) Consider the following “pay-as-you-go” social security system. The government sets $\tau^y_t = \tau$ for all $t$, and sets taxes for old consumers in each period so that the government budget balances.

i. Write down the government budget constraint, and determine $\tau^o_t$.

ii. Show how a change in $\tau$ affects the steady state capital/labor ratio.

iii. Show that it is possible, if $\tau$ is set correctly, to achieve the socially optimal steady state as a long-run competitive equilibrium.
(c) Consider a “fully-funded” social security system. Here, the government uses taxes on young consumers to finance the acquisition of government capital. That is, in period $t$ the government acquires $K^g_{t+1}$ units of capital, then in period $t + 1$ the government rents this capital to firms, and then converts it to consumption goods, and then transfers the proceeds lump sum in equal amounts to old consumers. We then have $K^{g}_t = L_t \tau^g_t$ and $K^g_{t+1}(1 + r_{t+1}) = -L_t \tau^o_{t+1}$. Suppose that $K^g_{t+1} = L_t g$ for all $t$, where $g$ is a constant.

i. Write down the government budget constraint.

ii. Show how a change in $g$ affects the steady state capital/labor ratio.

iii. Is it possible to set $g$ in the appropriate way to achieve the socially optimal steady state as a long-run competitive equilibrium? Why or why not?

iv. What do your results have to say about the efficacy of pay-as-you-go vs. fully-funded social security schemes?