

06E:204 Macroeconomics
Test 1 Solutions

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1. The answers to parts (a)-(c) are as follows:

(a) The government budget constraint is given by

$$g = tw(1 - l) \tag{1}$$

(b) A Pareto optimum is given by the solution to the social planner's problem, which is

$$\max_l u(1 - l - g, l),$$

and the first order condition for an optimum is

$$-u_1(1 - l - g, l) + u_2(1 - l - g, l) = 0, \tag{2}$$

with equation (2) solving for the Pareto optimal quantity of leisure. In a competitive equilibrium, the consumer solves the problem

$$\max_l u[w(1 - t)(1 - l), l],$$

and the first-order condition for an optimum is

$$-w(1 - t)u_1[w(1 - t)(1 - l), l] + u_2[w(1 - t)(1 - l), l] = 0. \tag{3}$$

The firm solves

$$\max_n (n - wn),$$

so the firm's demand for labor is infinitely elastic at $w = 1$, which implies that the competitive equilibrium real wage is $w = 1$. Therefore, substituting $w = 1$, and substituting for t from using (1) in (3), we obtain

$$-(1 - l - g)u_1(1 - l - g, l) + (1 - l)u_2(1 - l - g, l) = 0, \tag{4}$$

which solves for the competitive equilibrium quantity of leisure. Now, clearly (4) gives a different solution for l than does (2), so the competitive equilibrium is not Pareto optimal.

- (c) To determine how l responds to a change in g in a competitive equilibrium, totally differentiate (4) with respect to l and g to get

$$\frac{dl}{dg} = \frac{-u_1 + gu_{11} + (1-l)(-u_{11} + u_{12})}{u_1 - u_2 + g(-u_{11} + u_{12}) + (1-l)(u_{11} - 2u_{12} + u_{22})}. \quad (5)$$

Now, we can sign parts of the expression on the right hand side of (5). We know for example that $u_1 > 0$ (the utility function is increasing in both arguments), $u_{11} < 0$ (the utility function is strictly concave), $u_{11} - 2u_{12} + u_{22} < 0$ (the utility function is strictly concave), $-u_{11} + u_{12} > 0$ (leisure is a normal good), and $u_1 - u_2 < 0$ (from (4)), but the sign of $\frac{dl}{dg}$ is in general indeterminate. Even in the case where $g = 0$, so that $u_1 - u_2 = 0$, we get

$$\frac{dl}{dg} = \frac{-u_1 + (1-l)(-u_{11} + u_{12})}{(1-l)(u_{11} - 2u_{12} + u_{22})},$$

and we cannot sign the numerator. What is going on is that, when g increases there a negative income effect from the increase in government spending which will reduce the quantity of leisure consumed. However, an increase in g will also tend to increase the tax rate (though note that the tax rate also depends on the quantity of leisure through the government budget constraint), which reduces the after-tax real wage and implies that the consumer will substitute leisure for consumption. There will also be a further negative income effect on leisure from the increase in t .

2. The solutions to parts (a)-(d) are:

- (a) The social planner solves

$$\max_l [u(z(1-l)) + v(l) - s(\alpha(1-l))],$$

and the first-order condition for an optimum is

$$-zu'(z(1-l^*)) + v'(l^*) + \alpha s'(\alpha(1-l^*)) = 0, \quad (6)$$

which solves for the optimal quantity of leisure, l^* .

- (b) In a competitive equilibrium, the consumer solves

$$\max_l [u(w(1-l)) + v(l) - s(x)],$$

given w and x , and the first order condition for an optimum is

$$-wu'(w(1-l)) + v'(l) = 0. \quad (7)$$

From the firm's profit maximization problem, labor demand is infinitely elastic at the real wage $w = z$, so $w = z$ in equilibrium, which implies that

$$-zu'(z(1 - l^{**})) + v'(l^{**}) = 0$$

solves for the competitive equilibrium quantity of leisure, l^{**} . But then, since $s(\cdot)$ is an increasing function,

$$-zu'(z(1 - l^{**})) + v'(l^{**}) + \alpha s'(\alpha(1 - l^{**})) > 0,$$

and so $l^{**} \neq l^*$ (from (6)), and the competitive equilibrium is not Pareto optimal. Indeed, we can show that $l^{**} < l^*$ so that too much output (and too much pollution) is produced in the competitive equilibrium.

- (c) Now, when pollution is taxed at the rate t , and the consumer receives a lump sum transfer τ , the consumer's first-order condition for an optimum is

$$-wu'(w(1 - l) + \tau) + v'(l) = 0, \quad (8)$$

and the firm's profit maximization problem implies that labor demand is infinitely elastic at the real wage $w = z - t\alpha$, which is then the equilibrium real wage. Given the government's budget constraint, $\tau = t\alpha(1 - l)$ and the equilibrium real wage, if we substitute in (8) we obtain

$$-zu'(z(1 - l)) + v'(l) + t\alpha u'(z(1 - l)) = 0,$$

so if the government sets t such that

$$t = \frac{s'(\alpha(1 - l))}{u'(z(1 - l))},$$

then $l = l^*$ in a competitive equilibrium, so that the competitive equilibrium is Pareto optimal. Note that the tax serves to mimic a price system, since the appropriate tax is set equal to the rate at which the consumer is willing to trade off pollution for consumption at the margin. At the optimum, the firm and the consumer face the same tradeoffs at the margin, and the externality is internalized.

- (d) If there is a market for pollution rights, then the consumer solves

$$\max_{l,x} [u(w(1 - l) + px) + v(l) - s(x)],$$

and the two first-order conditions are

$$-wu'(w(1 - l) + px) + v'(l) = 0, \quad (9)$$

$$pu'(w(1-l) + px) - s'(x) = 0. \quad (10)$$

From the firm's profit maximization problem, labor demand is infinitely elastic at the real wage $w = z - p\alpha$, which is then the competitive equilibrium wage, so given this, (9), and (10), we obtain (6), and the competitive equilibrium is Pareto optimal, as the externality has been eliminated by completing the missing market.

3. In this model, the competitive equilibrium and the Pareto optimum are identical, and we can determine the Pareto optimum by solving a sequence of static problems. That is, the social planner solves

$$\max_{l_t} \log[\min(1 - l_t - g_t, \alpha l_t)],$$

and the solution to this problem is to set $1 - l_t - g_t = \alpha l_t$, which yields

$$\begin{aligned} l_t &= \frac{1 - g_t}{1 + \alpha}, \\ y_t &= n_t = \frac{\alpha + g_t}{1 + \alpha}, \\ c_t &= \frac{\alpha(1 - g_t)}{1 + \alpha}. \end{aligned}$$

Therefore, since leisure, output, employment, and consumption depend only on current government purchases, and government purchases follow a two-cycle, then leisure, output, employment, and consumption also follow a two-cycle. To determine interest rates, start with the consumer's problem, which is

$$\max_{\{c_t, l_t, s_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log[\min(c_t, \alpha l_t)]$$

subject to

$$c_t = w_t(1 - l_t) - s_{t+1} + (1 + r_t)s_t, \text{ for } t = 0, 1, 2, \dots$$

Now, the consumer will optimize by setting $c_t = \alpha l_t$, so this problem simplifies to

$$\max_{\{c_t, s_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$c_t \left(1 + \frac{w_t}{\alpha}\right) = -s_{t+1} + (1 + r_t)s_t, \text{ for } t = 0, 1, 2, \dots$$

Then, if we construct the consumer's intertemporal budget constraint, and then determine the first order conditions for an optimum, the marginal condition that we will need to determine the equilibrium real interest rate is

$$\frac{\beta c_t}{c_{t+1}} = \frac{1 + \frac{w_{t+1}}{\alpha}}{\left(1 + \frac{w_t}{\alpha}\right) (1 + r_{t+1})}$$

Now, since the competitive equilibrium real wage is $w_t = 1$ for all t , letting r^* denote the real interest rate in even periods and r^{**} the real interest rate in odd periods, then

$$\begin{aligned}\frac{\beta c^*}{c^{**}} &= \frac{1}{1 + r^*}, \\ \frac{\beta c^{**}}{c^*} &= \frac{1}{1 + r^{**}},\end{aligned}$$

and so the real interest rate also follows a two-cycle, and the real interest rate is higher in even periods than in odd periods. This pattern occurs because consumption is high in odd periods and low in even periods. In an attempt to smooth consumption, the consumer tries to borrow in even periods and save in odd periods, pushing interest rates up in even periods and down in odd periods, to the point where the consumer is reconciled to the choppy time path of consumption.