

06E:204 Macroeconomics
Test 2 Solutions

Steve Williamson
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1. The answers to parts (a)-(c) are as follows:

(a) The optimal steady state is the solution to

$$\max_{c^y, c^o, k} \min(c^y, c^o)$$

subject to

$$c^y + \frac{1}{1+n}c^o = k^\alpha - nk,$$

and the solution is

$$c^y = c^o = \frac{1+n}{2+n}[(k^*)^\alpha - nk^*]$$

$$k^* = \left(\frac{\alpha}{n}\right)^{\frac{1}{1-\alpha}}.$$

(b) For a consumer that pays a tax τ_t^y when young and a tax τ_{t+1}^o when old, optimal savings is given by

$$s_t = \frac{w_t - \tau_t^y + \tau_{t+1}^o}{2 + r_{t+1}}, \quad (1)$$

and so the equilibrium condition when all taxes are zero is given by

$$k_{t+1}(1+n) = s_t.$$

Then, from the first-order conditions for the firm's optimization problem, we have $w_t = (1-\alpha)k_t^\alpha$, and $r_{t+1} = \alpha k_{t+1}^{\alpha-1}$, so substituting in the equilibrium condition, we get

$$k_{t+1}(1+n) = \frac{(1-\alpha)k_t^\alpha}{2 + \alpha k_{t+1}^{\alpha-1}}, \quad (2)$$

which is a first-order difference equation which implicitly solves for $\{k_t\}_{t=1}^{\infty}$ given k_0 . Solving for the steady state, where $k_{t+1} = k_t = k^{**}$, we get, from (2),

$$k^{**}(1+n) = \frac{(1-\alpha)(k^{**})^\alpha}{2 + \alpha(k^{**})^{\alpha-1}},$$

and solving, we get

$$k^{**} = \left[\frac{1 - 2\alpha - \alpha n}{2(1+n)} \right]^{\frac{1}{1-\alpha}},$$

so clearly $k^{**} \neq k^*$, and the competitive equilibrium capital/labor ratio is different from what is optimal. You can show that there may be too little or too much capital in the steady state. In the competitive equilibrium, consumption of the young is

$$c^y = w_t - s_t = (1-\alpha)(k^{**})^\alpha - (1+n)k^{**},$$

and consumption of the old is

$$c^o = (1+r_{t+1})s_t = [k^{**} + \alpha(k^{**})^\alpha](1+n)$$

- (c) Finally, with the tax program in place, the government budget constraint is

$$L_t \tau_t^y + L_{t-1} \tau_t^o = 0,$$

so that $\tau_t^o = -(1+n)\tau_t^y$. The equilibrium condition, using the savings function (1), is then

$$k_{t+1}(1+n) = \frac{(1-\alpha)k_t^\alpha - \tau_t^y + \tau_{t+1}^o}{2 + \alpha k_{t+1}^{\alpha-1}}. \quad (3)$$

Therefore, if we set steady state taxes to achieve the optimal steady state, then steady state taxes are $\tau^o = -(1+n)\tau^y$, and the optimal tax rate, substituting in (3) solves

$$k^*(1+n) = \frac{(1-\alpha)(k^*)^\alpha - (2+n)\tau^y}{2+n},$$

or

$$\tau^y = \frac{(1-\alpha)(k^*)^\alpha}{2+n} - k^*(1+n)$$

Note that we could have $\tau^y > 0$, in which case this would be a standard social security scheme, with transfers from the young to the old, or we could have $\tau^y < 0$ with transfers from the old to the young at the optimum.

2. The answers to parts (a) and (b) are:

- (a) The Bellman equation for the dynamic programming problem associated with the social planner's problem is

$$v(k_t) = \max_{k_{t+1}} [u(\alpha k_t - k_{t+1}) + \beta v(k_{t+1})],$$

and the first order condition for an optimum is

$$-u'(\alpha k_t - k_{t+1}) + \beta v'(k_{t+1}) = 0,$$

with the envelope condition $v'(k_t) = \alpha u'(\alpha k_t - k_{t+1})$. Therefore, we get

$$-u'(c_t) + \alpha \beta u'(c_{t+1}) = 0,$$

which implies that, since $u(\cdot)$ is strictly concave, $c_{t+1} > c_t$ if $\alpha\beta > 1$, and there is unbounded growth if $\alpha\beta > 1$. That is, the interest rate in this economy in equilibrium is $\alpha - 1$, and if this rate of return is sufficiently large relative to the discount factor, then in equilibrium the representative consumer will choose to increase the capital stock forever. The fact that there is a constant marginal product of capital implies that there can be unbounded growth.

- (b) In the special case where $u(c) = \ln c$, guess that the value function takes the form $v(k_t) = A + B \ln k_t$, and use guess and verify methods to determine that $B = \frac{1}{1-\beta}$, and that the policy function is

$$k_{t+1} = \alpha\beta k_t,$$

with $c_t = \alpha(1 - \beta)k_t$. Then,

$$\frac{c_{t+1}}{c_t} = \frac{y_{t+1}}{y_t} = \frac{k_{t+1}}{k_t} = \alpha\beta,$$

so that consumption, output, the capital stock, and investment, all grow at the rate $\alpha\beta - 1$. Note that the rate of growth increases as the gross rate of return on capital, α , increases, and as the discount factor increases. A higher rate of return on capital lowers the price of future consumption in terms of current consumption, and the representative consumer saves more, which increases the growth rate. With a higher discount factor, the consumer is more patient, which again increases savings and causes an increase in the growth rate. Note that the savings rate here is β .

3. The resource constraint for this economy is

$$N_t c_t + K_{t+1} = \min(K_t, A_t N_t).$$

Now, if $K_{t+1} \geq A_{t+1} N_{t+1}$, then

$$N_t c_t \leq A_t N_t - A_{t+1} N_{t+1} < 0,$$

but consumption is nonnegative, so we must have $K_t < A_t N_t$ for $t = 1, 2, 3, \dots$. Then, if K_0^* is the initial capital stock, define $K_0 = \min(K_0^*, A_0 N_0)$, and the social planner's problem can be written as

$$\max \sum_{t=0}^{\infty} [\beta(1+n)]^t \frac{c_t^\gamma}{\gamma}$$

subject to

$$c_t + k_{t+1}(1+n) = k_t,$$

where $k_t \equiv \frac{K_t}{N_t}$, and given k_0 . Then, the Bellman equation associated with this problem is

$$v(k_t) = \max_{k_{t+1}} \left\{ \frac{[k_t - k_{t+1}(1+n)]^\gamma}{\gamma} + \beta(1+n)v(k_{t+1}) \right\}.$$

The first order condition for an optimum is

$$-(1+n)[k_t - k_{t+1}(1+n)]^{\gamma-1} + \beta(1+n)v'(k_{t+1}) = 0,$$

and the envelope condition is

$$v'(k_t) = [k_t - k_{t+1}(1+n)]^{\gamma-1}.$$

Then, it follows that

$$-c_t^{\gamma-1} + \beta c_{t+1}^{\gamma-1} = 0,$$

which implies that

$$\frac{c_{t+1}}{c_t} = \beta^{\frac{1}{1-\gamma}} < 1.$$

Therefore, in equilibrium per-capita consumption grows at a constant rate, which is negative. The economy does not grow because, given that capital and effective units of labor are perfect complements in production, growth in output is bounded by the slowest-growing factor of production, which will always be the capital stock. Since the rate of return to capital accumulation is zero, then given discounting by the consumer, the capital/labor ratio and per-capita consumption must fall over time.