

06E:204 Macroeconomics
Test 1 Solutions

Steve Williamson
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1. The answers to parts (a)-(c) are as follows:

(a) In a competitive equilibrium, the representative consumer solves

$$\max_l u(w(1-l), l),$$

and the first-order condition for an optimum is

$$-wu_1(w(1-l), l) + u_2(w(1-l), l) = 0. \quad (1)$$

The representative firm solves

$$\max_n \{[z(1-t) - w]n\},$$

which implies that the firm's demand for labor, n , is infinitely elastic at $w = z(1-t)$. In equilibrium, the government budget constraint is $g = tzn$, and given that $n = 1 - l$ in a competitive equilibrium, the tax rate t is determined by

$$t = \frac{g}{z(1-l)}. \quad (2)$$

Since the demand for labor is perfectly elastic at the real wage $w = z(1-t)$, this must be the equilibrium real wage, so substituting for t using (2), the equilibrium real wage is given by

$$w = z - \frac{g}{1-l}. \quad (3)$$

Then, substituting in equation (1) for w using (3), we obtain

$$-z - \frac{g}{1-l} u_1(z(1-l) - g, l) + u_2(z(1-l) - g, l) = 0, \quad (4)$$

and equation (4) solves for equilibrium l . Let l^* denote the quantity of leisure that solves equation (4). Then we can solve for output as $y^* = z(1-l^*)$, and for consumption from $c^* = y^* - g$. Finally, the equilibrium tax rate and real wage can be determined from equations (2) and (3) respectively, and employment is given by $n^* = 1 - l^*$.

- (b) When government spending is financed by a lump sum tax, we know that the competitive equilibrium is Pareto optimal. Thus, the representative consumer is at least as well off in this case as in part (a), and is better off if we can show that the allocation here is different from the one in part (a). Since the competitive equilibrium is Pareto optimal in this case, we can solve for competitive equilibrium quantities by solving the social planner's problem

$$\max_l u(z(1-l) - g, l),$$

and the first-order condition for an optimum is

$$-zu_1(z(1-l) - g, l) + u_2(z(1-l) - g, l) = 0, \quad (5)$$

which solves for the equilibrium quantity of leisure in this case, which we denote by l^{**} . Now, from (4), note that

$$-zu_1(z(1-l^*) - g, l^*) + u_2(z(1-l^*) - g, l^*) < 0,$$

so $l^* \neq l^{**}$, and so the consumer is better off with lump-sum taxation than with the proportional tax on output. This is because there is a distortion caused by the proportional tax - the consumer and the firm do not face the same effective real wage in equilibrium. As a result, there is a welfare loss due to taxation.

- (c) Rewriting equations (4) and (5) respectively, the equilibrium quantity of leisure in part (a) is determined by

$$z - \frac{g}{1-l} = \phi(l), \quad (6)$$

while the equilibrium quantity of leisure in part (b) is determined by

$$z = \phi(l), \quad (7)$$

where

$$\phi(l) = \frac{u_2(z(1-l) - g, l)}{u_1(z(1-l) - g, l)}. \quad (8)$$

Now, differentiating (8) with respect to l , we obtain

$$\phi'(l) = \frac{u_1(-zu_{12} + u_{22}) - u_2(-zu_{11} + u_{12})}{(u_1)^2} < 0,$$

assuming that consumption and leisure are normal goods. Therefore, since the left-hand side of equation (6) is less than z for all $l \in [0, 1]$, we must have $l^* > l^{**}$. This then implies that employment is higher when there is lump-sum taxation, output is higher, and consumption is higher. We have already shown that the real wage is higher with lump sum taxation than with proportional taxation of output. What is the intuition behind these results? With lump sum taxation, the consumer faces a higher real wage, which implies that, through positive income effects on consumption and leisure, consumption and leisure will tend to rise. However, the substitution effect of the increase in the real wage will be for consumption to rise and leisure to fall. There is also an additional effect, which is that the lump sum tax results in a pure negative income effect (relative to proportional taxation) on both consumption and leisure. On net, it is not clear from this reasoning whether consumption rises or falls, or whether leisure rises or falls. However, our analysis shows that leisure unambiguously falls and consumption unambiguously rises (assuming of course that consumption and leisure are normal goods).

2. Answers to parts (a) and (b) are as follows.

- (a) In this economy the competitive equilibrium is identical to the Pareto optimum, so we can solve for competitive equilibrium quantities by solving the social planner's problem, which is

$$\max_l [\ln(z(1-l) + k_0) + \beta \ln l]$$

Now, we cannot have a solution where $l = 0$, since $\ln l \rightarrow -\infty$ as $l \rightarrow 0$. The objective function in the above maximization problem is concave in l , so to determine whether or not there is a corner solution with $l = 1$, first differentiate with respect to l to obtain

$$\frac{\partial}{\partial l} [\ln(z(1-l) + k_0) + \beta \ln l] = \frac{-z}{z(1-l) + k_0} + \frac{\beta}{l}. \quad (9)$$

Then, if we evaluate this derivative at $l = 1$, we will have a corner solution to the optimization problem with $l = 1$ if and only if

$$\frac{-z}{k_0} + \beta \geq 0,$$

and an interior solution if and only if

$$\frac{-z}{k_0} + \beta < 0.$$

Thus, if $\frac{-z}{k_0} + \beta \geq 0$, then $l = 1$, $n = 0$, and $c = y = k_0$. If $\frac{-z}{k_0} + \beta < 0$, then there is an interior solution with $0 < l < 1$. Setting the first derivative in (9) to zero and solving for l , we obtain

$$l = \frac{\beta(z + k_0)}{z(1 + \beta)}. \quad (10)$$

Then, since $c = y = z(1 - l) + k_0$, we have

$$c = y = \frac{z + k_0}{1 + \beta}, \quad (11)$$

and given that $n = 1 - l$, we get

$$n = \frac{z - \beta k_0}{z(1 + \beta)}. \quad (12)$$

Now, to obtain equilibrium prices, first solve the firm's optimization problem, which is

$$\max_{n,k} (zn + k - wn - rk),$$

which implies that the firm's demand for labor, n , is perfectly elastic at $w = z$, and the demand for capital is perfectly elastic at $r = 1$. Since the supply of capital is perfectly inelastic in all cases (k_0 units of capital is supplied for any r), in all cases the equilibrium rental rate on capital is $r = 1$. The consumer will solve

$$\max_l [\ln(w(1 - l) + rk_0) + \beta \ln l],$$

which implies that the consumer chooses $l = 1$ if $w \leq \beta rk_0$, and there will be an interior solution otherwise. Therefore, the equilibrium real wage will be $w = z$ in the case where $\frac{-z}{k_0} + \beta < 0$ (and $0 < l < 1$), and $w \in [z, \beta k_0]$ otherwise. If the capital stock is sufficiently large and the marginal product of labor z is sufficiently small, then there are insufficient incentives for the consumer to supply any labor in equilibrium, and the consumer takes all his or her time as leisure.

- (b) If $\frac{-z}{k_0} + \beta < 0$, then from (10)-(12) it is clear that leisure falls, employment rises, consumption rises, and output rises when there is an increase in z . As well, the real wage ($w = z$) increases. There is no effect on the rental rate on capital. An increase in the marginal product of labor increases the real wage, which implies a positive income effect on both consumption and

leisure. However, there is a substitution effect, in that the relative price of leisure has risen, and this will cause the consumer to substitute away from consumption and towards leisure. On net consumption must rise, and for this utility function the substitution effect on leisure is larger than the income effect, so leisure falls and employment rises. If $\frac{-z}{k_0} + \beta \geq 0$, then there is no effect of z on equilibrium prices and quantities (other than that this affects the interval within which the real wage can fall), since neither the firm's nor the consumer's decisions will be affected at the margin.

3. Since the competitive equilibrium is Pareto optimal here, then given $\{g_t\}_{t=0}^{\infty}$ and $\{z_t\}_{t=0}^{\infty}$, we can solve for the competitive equilibrium sequence $\{l_t\}_{t=0}^{\infty}$ by solving the social planner's problem

$$\max_{\{l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(z_t(1-l_t) - g_t, l_t),$$

and so the following sequence of first-order conditions must hold in a competitive equilibrium.

$$-z_t u_1(z_t(1-l_t) - g_t, l_t) + u_2(z_t(1-l_t) - g_t, l_t) = 0, \quad (13)$$

for $t = 0, 1, 2, \dots$. Now, since the government sets government spending to hold private consumption constant at $c_t = c^*$ for all t , we must also have

$$z_t(1-l_t) - g_t = c^* \quad (14)$$

for $t = 0, 1, 2, \dots$. Here, equations (13) and (14) solve for l_t and g_t given z_t , so we can use these equations to do comparative statics in determining how l_t and g_t (and by implication output, employment, and the real interest rate) respond to a change in z_t . Totally differentiating equations (13) and (14), and arranging in matrix form,

$$\begin{array}{ccc} \begin{array}{c} z^2 u_{11} - 2z u_{12} + u_{22} \\ -z \end{array} & \begin{array}{c} z u_{11} - u_{12} \\ -1 \end{array} & \begin{array}{c} dl_t \\ dg_t \end{array} = \begin{array}{c} u_1 + z(1-l_t)u_{11} - (1-l_t)u_{12} \\ -(1-l_t) \end{array} dz_t \end{array}$$

Then, solving (I used Cramer's rule), we obtain

$$\frac{dl_t}{dz_t} = \frac{-u_1}{\nabla} < 0,$$

$$\frac{dg_t}{dz_t} = \frac{dy_t}{dz_t} = \frac{z u_1 + (1-l_t)(z u_{12} - u_{22})}{\nabla} > 0,$$

with

$$\nabla = zu_{12} - u_{22} > 0.$$

Note that, to sign all of the derivatives, I used the assumption $u_{12} > 0$, which implies that goods are normal. Thus, when z_t increases, consumption will tend to rise and leisure may increase or decrease due to opposing income and substitution effects. However, the government wants to hold consumption constant, which requires that it must increase g_t . This results in a negative income effect on consumption and leisure which is sufficient to cause leisure to increase. Output must increase as consumption is held constant while government purchases have increased. Since $w_t = z_t$ in equilibrium, the real wage rises when z_t rises. To determine the effects on the real interest rate, note that the consumer's optimization problem implies that the following first-order condition holds in equilibrium:

$$\frac{\beta u_1(c_{t+1}, l_{t+1})}{u_1(c_t, l_t)} = \frac{1}{1 + r_{t+1}},$$

but since $c_t = c^*$ in equilibrium, we have

$$r_{t+1} = \frac{u_1(c^*, l_t)}{\beta u_1(c^*, l_{t+1})} - 1 \quad (15)$$

for all t . Now, the results above imply that c_t , y_t , n_t , and w_t will be constant for $t = 0, 1, 2, \dots, T - 1$. Then, consumption remains constant for $t = T, T + 1, \dots$, while output increases, employment increases, and the real wage increases (note that government spending also increases) in period T and each of these variables then remains constant forever. From equation (15), we will have $r_{t+1} = \frac{1}{\beta} - 1$ for $t = 0, 1, \dots, T - 2, T, T + 1, \dots$, but for $t = T - 1$, we will have

$$r_T = \frac{u_1(c^*, l^*)}{\beta u_1(c^*, l^{**})} - 1,$$

where $l^* > l^{**}$. Therefore, since $u_{12} > 0$, $r_T > \frac{1}{\beta} - 1$, so that the real interest rate increases temporarily in period $T - 1$. In period $T - 1$ the consumer expects consumption to stay constant and leisure to fall. Since $u_{12} > 0$, consumption and leisure are complements, so the consumer will wish to increase consumption in period $T - 1$ and reduce it in future periods by borrowing in period $T - 1$. This tends to increase the real interest rate so as to make the consumer willing to accept the decrease in leisure.