06E:204 Macroeconomics
Test 2 Solutions

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1. The answers to parts (a)-(e) are as follows:

   (a) In a steady state, the resource constraint faced by the social planner is

   $$c^y + \frac{c^o}{1+n} = k^\alpha - nk, \tag{1}$$

   where c^y is consumption by the young, c^o is consumption by the old, and k is the capital/labor ratio. The socially optimal steady state allocation is obtained as the solution to the problem of solving

   $$\max_{c^y, c^o, k} (\ln c^y + \ln c^o)$$

   subject to (1). The optimal capital/labor ratio is given by the solution to

   $$\alpha k^{\alpha-1} - n = 0,$$

   or

   $$k = \left(\frac{\alpha}{n}\right)^{\frac{1}{1-\alpha}},$$

   and then if we define

   $$\phi = \left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha),$$

   the optimal consumption of the young and the old (solving the constrained optimization problem above) is given by

   $$c^y = \frac{\phi}{2};\ c^o = \frac{(1+n)\phi}{2}$$

   (b) The government's budget constraint is given by

   $$K^g_{t+1} = (1+r_t)K^g_t + L_t\tau^y_t + L_{t-1}\tau^o_t. \tag{2}$$

(c) A young consumer solves

$$\max_{s_t}\{\ln(w_t - s_t - \tau_t^y) + \ln[(1+r_{t+1})s_t - \tau_{t+1}^o]\}$$

The solution is

$$s_t = \frac{w_t - \tau_t^y}{2} + \frac{\tau_{t+1}^o}{2(1+r_{t+1})} \tag{3}$$

(d) To verify that the government budget constraint is met, substitute $\tau_t^o = -(1+n)\tau$, $\tau_t^y = \tau$, and $K_t^g = K_{t+1}^g = 0$ in equation (2) and show that it holds. In equilibrium, we have

$$k_{t+1}(1+n) = s_t, \tag{4}$$

and from the firm's profit maximization problem,

$$w_t = (1-\alpha)k_t^\alpha, \tag{5}$$

$$r_t = \alpha k_t^{\alpha-1}. \tag{6}$$

Then, substituting using (2), (5) and (6) in (4), we get

$$k_{t+1}(1+n) = \frac{(1-\alpha)k_t^\alpha}{2} - \frac{\tau}{2} - \frac{\tau(1+n)}{2(1+\alpha k_{t+1}^{\alpha-1})} \tag{7}$$

In the optimal steady state, we have $k_t = k_{t+1} = k = \left(\frac{\alpha}{n}\right)^{\frac{1}{1-\alpha}}$, and so we can substitute in (7) and solve for the tax $\tau = \tau^*$ that yields this as a competitive equilibrium. We get

$$\tau^* = \left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}\left[\frac{(1-\alpha)}{2} - (1+n)\left(\frac{n}{\alpha}\right)\right]$$

Note that we could have $\tau^* > 0$ or $\tau^* < 0$, as τ^* is a decreasing and continous function of n, and we have $\tau^* > 0$ for $n = 0$ and $\tau^* < 0$ for $n = \alpha$. The problem here is that there are no markets for intergenerational trade, but the government can accomplish intergenerational trade through the tax system. The appropriate lump sum transfers between young and old can achieve a socially optimal steady state. However, the optimal transfers could either flow from the young to the old or from the old to the young.

(e) Again, substitute $K^g_{t+1} = L_t\gamma$, $\tau^y_t = \gamma$, and $\tau^o_{t+1} = -\gamma(1 + r_{t+1})$ in the government budget constraint (2), and it is easy to show that this holds. In this case, the equilibrium condition is given by

$$K_{t+1} = L_t s_t + K^g_{t+1},$$

since the period $t+1$ capital stock is the quantity of capital accumulated by the young and by the government in period t. Rewriting the equilibrium condition in per-worker terms, we get

$$k_{t+1}(1 + n) = s_t + \gamma, \tag{8}$$

and from (3) we have

$$s_t = \frac{w_t}{2} - \gamma. \tag{9}$$

Note in particular here that a one-unit increase in γ, which is government capital accumulation per worker, is reflected in a one-unit decrease in capital accumulation by each worker. Then, using (5) and (9) to substitute in (8), we get

$$k_{t+1}(1 + n) = (1 - \alpha)k^\alpha_t. \tag{10}$$

Note from equation (10) that the sequence of equilibrium quantities of capital per worker $\{k_t\}^\infty_{t=0}$ is unaffected by γ. In general the steady state equilibrium allocation is not socially optimal, and changing γ will not affect this. This government policy is neutral due to Ricardian equivalance. A change in γ simply changes the timing of taxes for each young agent in each generation, without changing their lifetime wealth. As a result, the changes in government and private saving are exactly offsetting, and there is no effect on the equilibrium allocation.

2. The solutions to parts (a) and (b) are as follows.

(a) The competitive equilibrium in this economy is Pareto optimal, and we can solve the social planner's problem using dynamic programming. The state variable in the problem is k_t, the capital stock, and the value function is $v(k_t)$. The Bellman equation is given by

$$v(k_t) = \max_{k_{t+1}, l_t} \left\{ \ln[k^\alpha_t(1 - l_t)^{1-\alpha} - k_{t+1}] + \gamma \ln l_t + \beta v(k_{t+1}) \right\}.$$

Guess that the value function takes the form $v(k_t) = A + B \ln k_t$. Then, the first order conditions for the optimization problem on the right-hand side of the Bellman equation are

$$-\frac{1}{k_t^{\alpha}(1-l_t)^{1-\alpha}-k_{t+1}}+\frac{\beta B}{k_{t+1}}=0, \tag{11}$$

$$-\frac{(1-\alpha)k_t^{\alpha}(1-l_t)^{-\alpha}}{k_t^{\alpha}(1-l_t)^{1-\alpha}-k_{t+1}}+\frac{\gamma}{l_t}=0. \tag{12}$$

Then, solving (11) for k_{t+1} we get

$$k_{t+1}=\frac{\beta B}{1+\beta B}k_t^{\alpha}(1-l_t)^{1-\alpha}, \tag{13}$$

and substituting for k_{t+1} in equation (12) and solving for l_t gives

$$l_t=\frac{\gamma}{(1+\beta B)(1-\alpha)+\gamma}, \tag{14}$$

and so substituting in (13) using (14) gives

$$k_{t+1}=\frac{\beta B}{1+\beta B}k_t^{\alpha}\left[\frac{(1+\beta B)(1-\alpha)}{(1+\beta B)(1-\alpha)+\gamma}\right]^{1-\alpha}. \tag{15}$$

Next, plug in for l_t and k_{t+1} on the right-hand side of the Bellman equation using (14) and (15), and solve for A and B by equating coefficients on either side of the Bellman equation. The solution for B is

$$B=\frac{\alpha}{1-\alpha\beta},$$

and it is straightforward to verify that a unique solution exists for A. This then implies that the policy functions are

$$k_{t+1}=\alpha\beta k_t^{\alpha}\left[\frac{1-\alpha}{1-\alpha+\gamma(1-\alpha\beta)}\right]^{1-\alpha}, \tag{16}$$

$$l_t=\frac{\gamma(1-\alpha\beta)}{1-\alpha+\gamma(1-\alpha\beta)}, \tag{17}$$

with output given by

$$y_t=k_t^{\alpha}\left[\frac{1-\alpha}{1-\alpha+\gamma(1-\alpha\beta)}\right]^{1-\alpha}.$$

Note that leisure is a constant, i.e. even though the real wage will in general change over time, income and substitution effects cancel given log utility.

(b) If γ increases, this increases the weight on leisure in the utility function, and not surprisingly the consumer will consume more leisure in every period. Since work effort is lower, less output is produced given the capital stock, and investment will be lower. In the steady state, the capital stock is given by

$$k^* = \psi,$$

where

$$\psi = (\alpha\beta)^{\frac{1}{1-\alpha}} \left[\frac{1-\alpha}{1-\alpha+\gamma(1-\alpha\beta)} \right],$$

and the steady state quantity of output is

$$y^* = \psi^{\alpha}.$$

Thus, the steady state quantities of capital and output decrease as γ increases. Lower work effort implies that the marginal product of capital is smaller given the quantity of capital. Since the steady state is achieved when the capital stock is sufficiently large that the marginal product of capital falls to the point where further capital accumulation is no longer worthwhile, the steady state is achieved at a lower capital stock for higher γ.

3. The dynamic programming problem associated with the social planner's problem for this economy is

$$v(h_t) = \max_{c_t, h_{t+1}, u_t, \lambda_t, \mu_t} \left(\frac{c_t^{\gamma}}{\gamma} + \beta v(h_{t+1}) + \lambda_t(\alpha h_t u_t - c_t) + \mu_t[\delta h_t(1-u_t-v) - h_{t+1}] \right)$$

The following first-order conditions characterize an optimum:

$$c_t^{\gamma-1} - \lambda_t = 0,$$

$$\beta v'(h_{t+1}) - \mu_t = 0,$$

$$\alpha h_t u_t - c_t = 0,$$

$$\delta h_t(1-u_t-v) - h_{t+1} = 0,$$

and given that the objective function on the right-hand side of the Bellman equation is linear in u_t and that corner solutions for u_t cannot be optimal, we have

$$\lambda_t \alpha h_t - \mu_t \delta h_t = 0.$$

Here, we can follow the same solution procedure as in my notes to get the equilibrium gross growth rates for consumption, human capital, and output, and the solution for u_t :

$$\frac{c_{t+1}}{c_t} = \frac{h_{t+1}}{h_t} = \frac{y_{t+1}}{y_t} = [\beta\delta(1-v)]^{\frac{1}{1-\gamma}},$$

$$u_t = 1 - v - [\beta\delta^{\gamma}(1-v)]^{\frac{1}{1-\gamma}}.$$

Then, the time devoted to human capital accumulation is

$$1 - u_t - v = [\beta\delta^{\gamma}(1-v)]^{\frac{1}{1-\gamma}}$$

As the government chooses to allocate more of the consumer's time to producing goods for the government, the consumer substitutes away from time spent in human capital accumulation, which causes the growth rates of consumption, human capital, and output to fall. We also have

$$\frac{\partial u_t}{\partial v} = -1 + \frac{[\beta\delta^{\gamma}(1-v)]^{\frac{1}{1-\gamma}}}{(1-\gamma)(1-v)}.$$

While it is true that $[\beta\delta^{\gamma}(1-v)]^{\frac{1}{1-\gamma}} < 1-v$, since $u_t > 0$, we still cannot sign this expression, so it is not clear whether u_t increases or decreases.