Economics 501
Test 1 Solutions

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1. The solutions are as follows:

   (a) A competitive equilibrium is defined in the same way as in my notes, Chapter 1. This is an environment where the competitive equilibrium and the Pareto optimum are identical, so we can determine the equilibrium quantity of leisure by solving

$$\max_{l\in[0,1]} u\left[\alpha k_0 - (1-\alpha)g + 1 - l, l\right]. \tag{1}$$

If there is an interior solution (more about this later), then we can characterize a solution by the first-order condition

$$-u_1\left[\alpha k_0 - (1-\alpha)g + 1 - l, l\right] + u_2\left[\alpha k_0 - (1-\alpha)g + 1 - l, l\right] = 0. \tag{2}$$

Then, given the solution for l, we have

$$y = \alpha k_0 + \alpha g + 1 - l,$$

and

$$c = y - g = \alpha k_0 - (1-\alpha)g + 1 - l.$$

In a competitive equilibrium, the representative firm solves

$$\max_{n,k}\left[(\alpha - r)\,k + (1-w)n + \alpha g\right],$$

which implies that the firm's demands for capital and labor and perfectly elastic at $r = \alpha$ and $w = 1$, respectively, which implies $r = \alpha$ and $w = 1$ in equilibrium (whatever you do, don't differentiate the profit function with respect to k and n and then set these derivatives equal to zero, yielding $r = \alpha$ and $w = 1$; that is nonsense). An important point to note is that the firm will earn positive profits in equilibrium, equal to αg. Profits have to be consumed by someone, and that someone has to be the representative consumer, so it is convenient to assume that the profits are returned to the consumer (i.e. the consumer owns shares in the firm and receives the firm's profits as a dividend).

(b) To determine the effects of a change in g, totally differentiate (2) and solve to get

$$\frac{dl}{dg} = \frac{(1-\alpha)(-u_{11}+u_{12})}{u_{11}-2u_{12}+u_{22}},$$

and given that $\alpha < 1$, and assuming normal goods $(-u_{11}+u_{12} > 0)$ and a strictly concave utility function $(u_{11}-2u_{12}+u_{22} < 0)$, we have $\frac{dl}{dg} < 0$, i.e. there is a negative income effect of government spending on leisure, which is mitigated somewhat due to the fact that government spending is productive. Further, we have

$$\frac{dn}{dg} = \frac{(1-\alpha)(u_{11}-u_{12})}{u_{11}-2u_{12}+u_{22}} > 0,$$

$$\frac{dy}{dg} = \alpha + \frac{dn}{dg} = \frac{\alpha u_{11} - (1+\alpha)u_{12} + \alpha u_{22}}{u_{11}-2u_{12}+u_{22}} > 0,$$

$$\frac{dc}{dg} = -(1-\alpha) + \frac{dl}{dg} = \frac{(1-\alpha)(-u_{22}+u_{12})}{u_{11}-2u_{12}+u_{22}} > 0,$$

where we can sign the last derivative given strict concavity of the utility function and normal goods $(-u_{22}+u_{12} > 0)$.

(c) Now, to determine the optimal level of g, solve the following planner's problem:

$$\max_{l\in[0,1],g\geq 0} u\left[\alpha k_0 - (1-\alpha)g + 1 - l, l\right].$$

Note that the objective function is strictly decreasing in g, so the solution is $g = 0$. Government spending here is wasteful, and so there should be none of it at the optimum.

(d) (not a part of the question) Note that, if we go back to problem (1), that there could be a corner solution. That is, the solution to the problem could be $l = 1$, with

$$-u_1\left[\alpha k_0 - (1-\alpha)g, 1\right] + u_2\left[\alpha k_0 - (1-\alpha)g, 1\right] \geq 0.$$

In this case, leisure will not change when g changes, consumption falls by $1-\alpha$, and output increases by α for each unit increase in g.

2. The solutions are as follows:

(a) The production possibilities frontier is given by the relationship

$$c = z(l)(1-l), \tag{3}$$

and given the assumptions I have made, the PPF is strictly concave. When $l \geq l^*$, $c = 0$, and when $l = 0$, $c = \bar{z}$.

(b) To solve for a Pareto optimum, solve

$$\max_l u\left[z(l)(1-l), l\right],$$

so that the Pareto optimal quantity of leisure is $\hat{l}$, which solves

$$z(\hat{l}) - z'(\hat{l})(1-\hat{l}) = v(\hat{l}),$$

where the marginal rate of substitution function $v(l)$ is defined by

$$v(l) \equiv \frac{u_2\left[z(l)(1-l), l\right]}{u_1\left[z(l)(1-l), l\right]}.$$

Since there is an externality, we would not expect the competitive equilibrium to be the same as the Pareto optimum, so we need to work from the consumer problem and the firm problem to determine the competitive equilibrium. The consumer solves

$$\max_l u(w(1-l), l),$$

where w is the real wage, so the quantity of leisure chosen by the consumer is the solution to

$$w = v(l).$$

The firm's problem is

$$\max_n [z(l) - w]n,$$

treating l and w as given. Therefore, the firm's demand for labor is perfectly elastic at $w = z(l)$, and so the equilibrium real wage must be $w = z(l)$. Therefore, the competitive equilibrium quantity of leisure, $\tilde{l}$, is the solution to

$$z(\tilde{l}) = v(\tilde{l}).$$

Now, note that both the competitive equilibrium and the Pareto optimum are points on the production possibilities frontier defined by (3). Further, note that since $z'(\hat{l})(1-\hat{l}) < 0$, we have

$$z(\hat{l}) < v(\hat{l}), \tag{4}$$

so the Pareto optimal quantity of leisure does not satsify the condition required for a competitive equilibrium, so clearly the Pareto optimum and the competitive equilibrium are not the same, just as we suspected. Now, what is the relationship between the Pareto optimum and the competitive equilibrium? It seems intuitive that we should have $\hat{l} < \tilde{l}$, but we need to prove this. First, note that $z'(\hat{l}) < 0$, and $v'(\hat{l}) < 0$, so the result is not obvious. However, strict concavity of the utility function and (4) guarantee that $z(l) < v(l)$ for all $l < \hat{l}$ (a picture helps here, which I'll show you in class). Therefore, it must be the case that $\tilde{l} > \hat{l}$, and we can then conclude that consumption (equal to output) is higher at the Pareto optimum than at the competitive equilibrium. The consumer fails to take account of the negative externality and therefore consumes too much leisure in equilibrium, and too little output is produced.

3. The solutions are as follows:

   (a) See my notes, Chapter 1.

   (b) See my notes, Chapter 1, and set $g_t = 0$ for all t.

   (c) In this problem, the competitive equilibrium is Pareto optimal, so we can solve for equilibrium quantities from the planner's problem which is just an infinite sequence of static problems. That is, the planner solves

$$\max_{l_t}[\alpha \log(z_t(1-l_t)) + \log l_t]$$

and the solution is

$$l_t = \frac{1}{1+\alpha},$$

$$n_t = \frac{\alpha}{1+\alpha},$$

$$c_t = y_t = \frac{z_t \alpha}{1+\alpha}.$$

To solve for prices, note that the consumer equates his or her marginal rate of substitution leisure for consumption to the real wage rate, which implies that

$$\frac{c_t}{\alpha l_t} = w_t,$$

and so in equilibrium $w_t = z_t$. As well, the consumer equates the intertemporal marginal rate of substitution for consumption to the inverse of the gross real interest rate, or

$$\frac{\beta c_t}{c_{t+1}} = \frac{1}{1+r_{t+1}},$$

so

$$r_{t+1} = \frac{z_{t+1}}{\beta z_t} - 1.$$

Therefore, given the path for productivity, the real interest rate the consumer faces (recall the consumer faces r_{t+1} in period t) is $\frac{1}{\beta} - 1$, or the rate of time preference, for $t = 1, 2, 3, ...T-2$, and for $t = T+1, T+2, ...$. For period $T-1$, the consumer faces a real rate of

$$r_T = \frac{\gamma}{\beta} - 1 > \frac{1}{\beta} - 1.$$

In period T, the consumer faces a real rate of

$$r_{T+1} = \frac{1}{\gamma\beta} - 1 < \frac{1}{\beta} - 1.$$

That is, the consumer likes to smooth consumption over time, and in period $T-1$ anticipates an increase in consumption and therefore wishes to borrow in order to smooth consumption over time. Of course it is technologically impossible to transfer consumption from one period to another in this economy, so the interest rate must increase in period $T-1$ to reconcile the consumer to the consumption path he or she will be stuck with. Similarly, in period T the consumer anticipates that consumption will fall next period, and therefore wishes to save in order to smooth consumption over time. In this case the interest rate must fall to make saving unattractive for the consumer and reconcile him or her to the decrease in consumption which must occur.

(d) Since the Ricardian equivalence theorem holds in this environment, and we are not changing the present value of government spending (which equals zero), this has no effect on the competitive equilibrium quantities or prices, except for that there exactly offsetting changes in the consumer's saving and government saving in each period.The governent's intertemporal budget constraint holds in equilibrium, so we will have

$$\tau_0 + \sum_{t=1}^{\infty} \frac{\tau_t}{\prod_{i=1}^{t}(1+r_i)} = 0,$$

or, given the equilibrium real interest rates we calculated above,

$$\sum_{t=0}^{T-1} \beta^t \tau^{**} + \frac{\beta^T}{\gamma}\tau^* + \sum_{t=T+1}^{\infty} \beta^t \tau^{**},$$

which allows us to solve for τ^{**} in terms of τ^*, using the formula for the sum of an infinite geometric series, i.e.

$$\tau^{**} = \frac{\frac{-\tau^*\beta^T}{\gamma}}{\frac{1}{1-\beta} - \beta^T} < 0.$$

That is, if the government runs a surplus during period T, and it sets taxes at the same level for all other periods, then in all other periods it must run a deficit to meet the intertemporal government budget constraint.