

Economics 501, Macroeconomics I
Test 2 Solution

STEVE WILLIAMSON

Tuesday, November 14, 2006

1. Solutions are as follows.

- (a) Clearly it must be optimal for the young in each period to consume zero, so the aggregate resource constraint for this economy is given by

$$F(K_t, L_t) + (1 - \delta)K_t = K_{t+1} + L_{t-1}c_t^o,$$

for $t = 0, 1, 2, \dots$, with K_0 given, which we can write as

$$f(k_t) + (1 - \delta)k_t = k_{t+1}(1 + n) + \frac{c_t^o}{1 + n}.$$

A planner who could choose among steady states, where $k_t = k$ and $c_t^o = c^o$ for all t , solves

$$\max_k u[\{(1 + n)[f(k) - (n + \delta)k]\}],$$

and since $u(\cdot)$ is a strictly increasing function, the optimal steady state is characterized by the steady state capital stock k^* that satisfies

$$f'(k^*) - \delta = n, \tag{1}$$

i.e. at the optimum the net marginal product of capital equals the population growth rate.

- (b) The government's budget constraint is

$$L_t\tau_t = L_{t-1}\phi_t,$$

or

$$\tau_t = \frac{\phi_t}{1 + n}. \tag{2}$$

- (c) Since consumers receive no utility from consumption when young, their savings when young is equal to their disposable income, or

$$s_t = w_t - \tau_t$$

and when old they rent out the capital they acquired when young at the capital rental rate r_t , and then sell the remaining capital. We need to take account of depreciation here. There are two ways to do this. Either assume an old consumer rents one unit of capital to the firm, and receives r_t units of capital in return and what is left of the capital after depreciation, i.e. $1 - \delta$ units, or that the consumer rents a unit of capital in exchange for r_t and receives one unit of capital back from the firm after the rental. As long as you are consistent in how the exchange is modeled from the firm's side, either approach gives the same result (though r_t will be different). Taking the first approach, consumption in old age for a consumer born in period t is given by

$$c_{t+1}^o = (1 - \delta + r_{t+1})(w_t - \tau_t) + \phi_{t+1}.$$

- (d) The capital that is rented to firms in period $t + 1$ is just $L_t s_t$, the total savings of the young from the previous period, and the rental demand for capital from firms is K_{t+1} in period $t + 1$, so market-clearing in the capital rental market each period gives

$$K_{t+1} = L_t s_t,$$

or

$$k_{t+1}(1+n) = w_t - \tau_t. \quad (3)$$

From the firm's optimization problem, in equilibrium we must have $f'(k_t) = r_t$ and $f(k_t) - k_t f'(k_t) = w_t$, so substituting in (3) and simplifying, we get

$$k_{t+1} = \frac{1}{1+n} [f(k_t) - k_t f'(k_t) - \tau_t], \quad (4)$$

which is a difference equation that solves for the sequence of competitive equilibrium capital labor ratios $\{k_t\}_{t=1}^{\infty}$ given k_0 .

- (e) Now, if the government sets $\tau_t = \tau$ and $\phi_t = \phi$ for all t , then from (2),

$$\phi = (1+n)\tau. \quad (5)$$

What we have set up is a social security system, and n is the rate of return that consumers receive through the system. For a given τ , the steady state quantity of capital k^{**} , from (4), is the solution to

$$k^{**} = \frac{1}{1+n} [f(k^{**}) - k^{**} f'(k^{**}) - \tau] \quad (6)$$

To find the level of τ that implies a steady state competitive equilibrium with $k^{**} = k^*$, since $f'(k^*) = n + \delta$, and letting τ^* denote the optimal tax on the young, from (6) we obtain

$$k^* = \frac{1}{1+n} [f(k^*) - k^*(n + \delta) - \tau^*]$$

or, solving for τ^* ,

$$\tau^* = f(k^*) - k^*(1 + 2n + \delta) \quad (7)$$

Then, an optimal policy satisfies (7) and the following two equations:

$$\phi^* = (1+n)\tau^*$$

$$f'(k^*) = n + \delta$$

- (f) Government policy here is a social security program that transfers wealth across generations. Because of a fundamental market incompleteness, competitive markets do not necessarily accomplish efficient intergenerational redistribution, and the government can improve on the competitive equilibrium outcome. Note that it is possible that $\tau^* < 0$, in which case wealth would be transferred from the old to the young each period at the optimum.

2. The solutions are as follows:

- (a) The competitive equilibrium and the Pareto optimum are the same allocation here, so to solve for competitive equilibrium quantities we can solve the social planner's problem, letting $v_e(k_t, z_t)$ denote the value function in even periods and $v_o(k_t, z_t)$ the value function in odd periods. There are two Bellman equations, which are

$$v_e(k_t, z_t) = \max_{k_{t+1}} [\theta^* \ln(z_t k_t - k_{t+1}) + \beta E_t v_o(k_{t+1}, z_{t+1})]$$

$$v_o(k_t, z_t) = \max_{k_{t+1}} [\theta^{**} \ln(z_t k_t - k_{t+1}) + \beta E_t v_e(k_{t+1}, z_{t+1})]$$

- (b) Guess that the value functions are

$$v_e(k_t, z_t) = A_e + B_e \ln k_t + D_e \ln z_t,$$

$$v_o(k_t, z_t) = A_o + B_o \ln k_t + D_o \ln z_t.$$

where the coefficients in the value functions need to be determined. Next, given your guesses, solve the optimization problems on the right-hand sides of each Bellman equation, and then equate coefficients on either side of the Bellman equations. It is straightforward to show that the guesses concerning the value functions are correct, and that the optimal decision rules for investment ($= k_{t+1}$ here because there is 100% depreciation) are

$$k_{t+1} = \frac{\beta B_o z_t k_t}{\theta^* + \beta B_o}, \text{ for } t \text{ even,}$$

$$k_{t+1} = \frac{\beta B_e z_t k_t}{\theta^{**} + \beta B_e}, \text{ for } t \text{ odd,}$$

and B_o and B_e solve

$$B_e = \theta^* + \beta B_o,$$

$$B_o = \theta^{**} + \beta B_e.$$

Then, solving for B_o and B_e , and substituting back into the optimal decision rules, we obtain

$$k_{t+1} = \frac{\beta(\theta^{**} + \beta\theta^*)z_t k_t}{\theta^* + \beta\theta^{**}}, \text{ for } t \text{ even,}$$

$$k_{t+1} = \frac{\beta(\theta^* + \beta\theta^{**})z_t k_t}{\theta^{**} + \beta\theta^*}, \text{ for } t \text{ odd.}$$

Since $c_t + k_{t+1} = z_t k_t$, we also get

$$c_t = \frac{(1 - \beta^2)\theta^* z_t k_t}{\theta^* + \beta\theta^{**}}, \text{ for } t \text{ even,}$$

$$c_t = \frac{(1 - \beta^2)\theta^{**} z_t k_t}{\theta^{**} + \beta\theta^*}, \text{ for } t \text{ odd.}$$

You can also solve for the other undetermined coefficients, to verify that you actually guessed correctly concerning the functional forms for the value functions.

- (c) In even periods, investment is low, in anticipation of the fact that the marginal utility of consumption will be low next period, while current consumption is high because the current marginal utility of consumption is high. However, in odd periods, investment is high and consumption is low. In response to the “seasonal” fluctuations in the marginal utility of consumption, consumption and investment fluctuate in a predictable fashion. In response to unpredictable technology shocks, however, there is consumption smoothing. High z_t implies that current consumption is high and current investment is high as well, so that future consumption will be higher.
- (d) For t even,

$$E_t[\ln y_{t+1} - \ln y_t] = \mu + \ln \beta + \ln \left(\frac{\theta^{**} + \beta\theta^*}{\theta^* + \beta\theta^{**}} \right),$$

and for t odd,

$$E_t[\ln y_{t+1} - \ln y_t] = \mu + \ln \beta + \ln \left(\frac{\theta^* + \beta\theta^{**}}{\theta^{**} + \beta\theta^*} \right),$$

where $\mu = E_t \ln z_{t+1}$. Therefore the expected growth rate in output in even periods is lower than in odd periods, and anticipated growth rates are independent of the current productivity shock, depending only on the mean of the log of next period’s technology shock, β , θ^* , and θ^{**} . Output is expected to grow at a higher rate in odd periods, when investment is high, and at a lower rate in even periods, when investment is low. This is a simple theory of seasonal cycles, driven by preference shocks, and business cycles, driven by technology shocks.