Liquidity and market participation

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A model of market participation in competitive asset markets is constructed. There are two assets, a liquid asset traded without cost, and an illiquid asset traded subject to fixed transactions costs. Agents have random needs for liquidity, but they are precluded from trading claims contingent on their type, and from diversifying between assets. Depending on parameter values, both assets or only one asset may be traded in equilibrium. There exists a participation externality, in that agents do not take account of the effect of their participation decisions on asset prices. There tends to be an underprovision of liquidity in equilibrium.

Key words: Liquidity; Market participation; Asset markets
JEL classification: G1; G2

1. Introduction

The purpose of this paper is to study the behavior of asset markets, in the context of a model where agents have limited ability to insure against random liquidity needs. There is an absence of contingent claims markets, and fixed transactions costs limit participation in assets markets. We examine the determinants of the volume of trade and asset prices in markets for liquid and illiquid assets. In particular, we are interested in the degree of participation in the two asset markets, and in whether there exists a tendency for underprovision or overprovision of liquidity (i.e., too little or too much participation in the market for the liquid asset relative to the illiquid asset market).

Liquidity refers to the ease with which an asset can be bought or sold. Though this definition of liquidity is straightforward, capturing liquidity in economic

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models can be challenging. Modeling liquidity requires the modeling of a trading friction, and there are at least three approaches which have been taken in the literature. First, in search/matching models of money, such as Kiyotaki and Wright (1989), heterogeneity and spatial separation imply that it takes time to trade an object for the object one wishes to acquire. The liquidity of an object is then reflected in its general acceptability. Second, illiquidity can be modeled as a private information friction, as in the classic 'lemons' model of Akerlof (1971) or in Williamson (1988, 1992). In these approaches, problems of adverse selection or moral hazard can make it difficult or impossible to trade a particular asset, even if the asset is known by the seller to be of good quality. Third, it can simply be posited that asset purchases or sales are subject to transactions costs, as in Aiyagari and Gertler (1991), Chatterjee and Corbae (1992), and Greenwood and Williamson (1989). This paper takes the latter approach, in that the degree of liquidity of an asset decreases as the costs incurred in buying and selling it increase.

An issue which has received much attention in the literature on asset liquidity is whether or not unfettered markets provide the appropriate mix of liquid and illiquid assets. For example, monetary models are often used to address questions associated with the optimal quantity of money [see Woodford (1990) for a survey]. Many of these models exhibit a monetary externality; in a competitive equilibrium economic agents tend to economize more than is socially optimal on their holdings of cash balances, and there is an underprovision of liquidity. The banking model of Diamond and Dybvig (1983) is another example of an environment where too little liquidity can be supplied in equilibrium. In the Diamond–Dybvig model, the underlying assets are illiquid, but a bank can transform these assets into liquid ones and thus insure agents against random needs for liquidity. Given the structure of deposit contracts, there may exist a bank run equilibrium in which the banking system fails to supply enough liquidity.

This paper, in addition to touching on issues studied in the literature on liquidity provision, is closely related to the market participation literature, for example Chatterjee (1988), Pagano (1989a, b), and Allen and Gale (1991). In these models, agents typically have the choice between a riskless activity and a risky activity, with fixed costs which prevent participation in both activities and an absence of markets for sharing risk. More participation in the risky activity makes the activity less risky for each individual and, given that agents are risk-averse, this encourages more participation. Thus, there is a strategic complementarity which can produce multiple equilibria. These market participation models have been used to address the issue of provision of liquidity. For example, Pagano (1989a) studies a model of asset markets where agents have a choice between participating in a market for a liquid (riskless) asset or in a market for risky equity. There may exist Pareto-ranked equilibria, where agents are better off in the equilibria with greater participation in the risky
equity market. In an equilibrium with a low level of equity market participation, there is then an overprovision of liquidity.

The model constructed here has three periods. As in Diamond and Dybvig (1983), the agents in the model have uncertain preferences; in the first period they do not know whether they will be early consumers (receive utility from consumption in the middle period) or late consumers (receive utility from consumption in the final period). The preference shocks in the model are intended to capture uncertain needs for liquidity, and the resulting risk is assumed to be uninsurable. There are two assets in the model, a liquid asset and an illiquid asset. Assets are acquired in the first period, and in the final period they pay off a certain return per unit invested. Agents are able to self-insure against the need for liquidity, as both assets can be traded on a competitive asset market in the middle period. However, an agent incurs a fixed cost if she buys or sells the illiquid asset, while there is costless trading in the liquid asset. Thus, if the prices of the liquid and illiquid assets were the same, the liquid asset would provide better insurance against the need for liquidity. Also, holding transactions costs constant, the higher the price of an asset in the middle period, the better the liquidity service it provides. In the first period, agents are limited as to which asset markets they can participate in. That is, they cannot hold both the liquid and illiquid assets. However, they may participate in both assets markets (if they choose) in the middle period.

As in Chatterjee (1988), Pagano (1989a, b), and Allen and Gale (1991), there is a focus here on the determinants of market participation in an environment where there are some impediments to participating in all markets, and an absence of complete contingent claims markets. However, in other ways this model is quite different. First, in Chatterjee (1988), Pagano (1989a, b), and Allen and Gale (1991), more participation in the risky activity permits better risk sharing, which is the source of a strategic complementarity. Here, the source of idiosyncratic risk is individual preference shocks, and market participation affects risk sharing through its effects on asset prices. Given the absence of a strategic complementarity in our model, the competitive equilibrium is unique. Rather than focusing on issues of coordination here, our interest is in the effects of market participation on asset prices, and how this affects agents’ ability to insure against the need for liquidity.¹ Second, our notion of liquidity differs. For example, Pagano (1989b) studies liquidity in a strategic context, where an asset is defined to be more liquid if its price is less volatile and will move less adversely if a given quantity is sold. Endogenous participation, which affects volatility and the responsiveness of price to quantity, then makes the degree of liquidity endogenous. Here, liquidity is defined more conventionally in terms of the

¹The model of Allen and Gale (1991) has preference shocks, but the mechanism by which market participation affects risk-sharing is similar to what is studied in Chatterjee (1988) and Pagano (1989a, b).
intrinsic properties of assets, i.e., the size of the costs incurred in buying or selling the asset.\(^2\) Third, Chatterjee (1988), Pagano (1989a, b), and Allen and Gale (1991) all construct models with heterogeneous agents, and the agents participating in different markets have different characteristics, in general. In the model constructed here, agents are identical, ex ante, but different agents may make different participation decisions in equilibrium. Fixed transactions costs introduce nonconvexities into each agent's optimization problem, and this in turn yields equilibria where agents who are identical in the first period may make different market participation choices.

Depending on parameter values, the competitive equilibrium may have the property that only the liquid or illiquid asset is traded, or that both assets are traded. As intuition might tell us, an increase in the costs of exchanging the illiquid asset implies that more liquid assets are held, and a decrease in the rate of return on the illiquid asset leads to less participation in the illiquid asset market.

Since this is a model with incomplete contingent claims markets, and given what is known about environments with incomplete markets [e.g., Hart (1975) and Geanakoplos and Polemarchakis (1986)], we would not expect a competitive equilibrium to be Pareto optimal, in general. Using a notion of constrained Pareto optimality adapted to this environment, we derive conditions under which competitive equilibria are suboptimal. That is, liquidity is underprovided and there is too much participation in the market for the illiquid asset, relative to the social optimum. If we interpret the risky activity in the 'coordination failure' models of Chatterjee (1988), Pagano (1989a), or Allen and Gale (1991) as playing a role similar to the illiquid asset in our model, then our results are quite different. The coordination failure models deliver the result that there exist multiple Pareto-ranked equilibria, and high participation (in the risky activity) equilibria are preferred. That is, there tends to be overprovision of liquidity in these models.

In our model, if only the liquid asset is traded in equilibrium, and we consider a small parameter change (such as an increase in the payoff to the illiquid asset in the final period) which would cause some agents to acquire illiquid assets in the initial period, a larger quantity of the middle-period endowment is absorbed in the form of transactions costs, and both asset prices tend to fall. The decrease in asset prices lessens the ability of agents to self-insure against the need for liquidity. In fact, agents would have been better off if they had continued to participate only in the market for the liquid asset. Due to the absence of a complete set of contingent claims markets, there exists a market participation externality; agents do not take account of the effects of their market participa-

\(^2\)In some sense market participation affects liquidity here, as is the case in Pagano's work. However, participation affects liquidity only through the levels of market prices, rather than through price volatility.
tion decisions on asset prices. The existence of this externality tends to lead to overparticipation in the market for the illiquid asset.

Some numerical examples are used to illustrate the model's properties. It is shown that it is possible to support trade in the illiquid asset even with relatively large fixed transactions costs. It is also possible to have equilibria exhibiting trade in both assets over a wide range of transactions costs. If the probability of consuming early (requiring liquidity) is low, then competitive equilibria tend to be constrained Pareto optimal. If the probability of consuming early is high, there can exist a wide range of parameter values for which the illiquid asset is traded (and sometimes only the illiquid asset is traded), but a Pareto improvement would occur if trade in the illiquid asset were completely shut down and all agents traded liquid assets. Thus, if we view this as a model describing when innovation occurs in asset markets, there is a tendency for too much innovation. 

When technological innovation causes transactions costs to fall, there is a tendency for more trade in the illiquid asset and less trade in the liquid asset. However, the problem here is that there is a tendency for introduction of the illiquid asset when the level of transactions costs is too high relative to the optimum.

The remainder of the paper is organized as follows. In section 2, the model is constructed, and agents' optimization problems are discussed in section 3. In section 4, a competitive equilibrium is defined, the candidate equilibria are determined, and existence and uniqueness of equilibrium is established. Section 5 contains some general results on the optimal provision of liquidity in the model, and section 6 gives a discussion of the numerical examples. The final section is a conclusion.

2. The model

There are three periods (0, 1, and 2) and a continuum of agents with unit mass. There are two types of agents. A type 1 agent consumes in period 1 and has preferences given by \( u(c_1) \), and a type 2 agent consumes in period 2 with utility \( v(c_2) \), where \( c_i \) denotes consumption in period \( i \). Type 1 agents will be denoted early consumers and type 2 agents late consumers.\(^3\) The fraction of early consumers in the population is \( \pi \), where \( 0 < \pi < 1 \). Here, \( \pi \) is public knowledge in period 0, but an agent does not learn her type until period 1. Thus, \( \pi \) is the probability of consuming early for any agent. It is assumed that \( u(\cdot) \) and \( v(\cdot) \) are increasing and concave, that \( v'(0) = u'(0) = \infty \), and that \(- cu''(c)/u'(c) \geq 1\).\(^3\)

\(^3\)The qualitative predictions of the model should not change if agents have preferences \( u(c_1) + \delta u(c_2) \), where \( \delta \) is the discount factor, and \( \delta \) differs across agents but is unknown at date 0. The extreme assumption that agents either get zero utility from period 1 consumption or from period 2 consumption lends tractability to the problem.
In period 0, each agent receives an endowment of \( x_0 \) units of an investment good, which can be used to produce either of two assets, denoted 1 and 2. Asset 1 yields a return of \( \beta_1 \) units of the consumption good in period 2 for each unit invested in period 0. Late consumers each receive an endowment of \( x_1 \) units of the consumption good in period 1, while early consumers receive nothing at that time. The fact that only late consumers receive an endowment of the consumption good in period 1 maximizes the potential gains from trade.

As in the model of Diamond and Dybvig (1983), agents face preference shocks which are meant to stand in for random liquidity needs. Agents would like to arrange in period 0 to make contingent trades of assets for consumption goods (if an early consumer) or trades of consumption goods for assets (if a late consumer) in period 1. However, there are three elements of the model that limit agents' ability to insure against preference shocks. First, markets are incomplete, in that agents are precluded from trading with one another in period 0 (note that this restriction prevents agents from setting up a Diamond–Dybvig 'bank'). Second, an agent is restricted from diversifying in period 0; her initial portfolio consists entirely of asset 1 or asset 2. This diversification restriction could result from fixed costs to committing funds to a particular asset market. If these fixed costs are sufficiently large, an agent will always hold only one asset in her portfolio. Third, assets are traded subject to transactions costs in period 1. Trade in the market for asset 1 (the liquid asset) is costless. However, if an agent trades asset 2 (the illiquid asset) for consumption goods in period 1, she incurs a fixed cost of \( z_1 \) units of the consumption good. Similarly, an agent exchanging consumption goods for the illiquid asset in period 1 incurs a fixed cost \( z_2 \). It is convenient to abstract from intermediation arrangements here. That is, the environment is such that agents cannot share the fixed transactions costs through an intermediary. The fixed transactions costs can be taken to represent the residual costs that individuals must bear after intermediation has served to economize on other costs. Note that transactions costs are real resource costs in the model; when illiquid assets are bought and sold, there is less of the consumption good available in the aggregate.

The model has at least two interpretations, the first being as a model of participation in an equity market. Here, equity (the illiquid asset) is costly to trade, and thus will be inferior to the liquid asset for early consumers if both assets trade at the same price. However, equity is superior to the liquid asset for late consumers if it bears a higher two-period rate of return. In the context of equity trading, there are fixed costs associated with acquiring the expertise to participate in the market for equities, and in thick markets the cost in terms of time and effort of trading a large quantity is similar to the cost of trading a small quantity. There is much empirical evidence which supports the notion that there are significant fixed transactions costs in equities markets. For example, King and Leape (1984) and Mankiw and Zeldes (1990) find evidence that only a small fraction of consumers hold stocks, and evidence in Blume, Crockett, and Friend
(1974) and Blume and Friend (1978) supports the view that, among consumers who hold stocks, the number of stocks held is very few, on average. If fixed costs were insignificant in the equities market, we would tend to observe most of the population holding a well-diversified portfolio of stocks.

The second interpretation of the model is in terms of banking and currency-holding. Here, we can suppose that banks hold the illiquid asset and that their only role is to facilitate transactions, with consumers holding illiquid assets indirectly by acquiring bank deposits. The sale or purchase of the illiquid asset is an accounting transaction which requires that both parties to the transaction incur costs of communicating with the bank (through check clearing, for example). In this interpretation, the liquid asset is currency, and it is thus assumed to have a terminal value of \( \beta_1 \) per unit invested in period 0. It would be straightforward to embed the model in an overlapping generations structure, in such a way that currency would have no intrinsic value, but could have value in equilibrium,\(^4\) and the model would have essentially the same implications.

There are similarities between this model and the banking model constructed by Diamond and Dybvig (1983), in that consumers have uncertain preferences and uncertain demands for liquidity. An important difference is that this model precludes trading among agents in the first period, while Diamond and Dybvig's model closes off communication in the middle period, through a sequential service constraint [see Wallace (1988) for an explicit treatment of sequential service]. Also, there is no opportunity in this model, where there is in Diamond and Dybvig's, for one-period storage of the consumption good. The fact that we have closed down a market (the market for insurance against the need for liquidity) gives this model something in common with approaches in the incomplete markets literature [e.g., Hart (1975) and Geanakoplos and Polémarchakis (1986)]. In contrast though, part of the market structure is determined endogenously here. That is, the asset trading technology, in part, will determine what assets are traded in equilibrium. It may be possible to have equilibria where both assets are traded, or where only the liquid or illiquid asset is traded. Note that, if there were no transactions costs here, then (with or without the restriction on diversification across assets in period 0) only the asset with the higher two-period rate of return would be held in equilibrium.\(^5\)

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\(^4\)To do this, collapse periods 0 and 1 into the first period of each agent's life. If an agent invests in the liquid asset, she buys fiat money at the beginning of the first period of life from the old of the previous generation. Asset trading takes place between young early consumers and young late consumers at the end of each period, after the current old die.

\(^5\)This model and related Diamond–Dybvig type models are more than just special cases of the standard incomplete markets model. To model liquidity and illiquidity, more is needed than incomplete markets.
3. Optimization

Let \( q_1 (q_2) \) denote the price of the liquid (illiquid) asset in terms of the period 1 consumption good. An agent's sequence of decisions is represented in the decision tree in fig. 1. In period 0, an agent chooses whether to acquire the liquid or the illiquid asset with her endowment of \( x_0 \) units of the investment good. In period 1, if the consumer learns she is an early consumer, she then sells her stock of assets, provided transactions costs do not exceed the proceeds from the asset sale, and consumes. If she is a late consumer, the agent enters period 1 with a stock of assets and an endowment of the consumption good, \( x_1 \). Since she will not consume until period 2, the consumption endowment is exchanged for claims to assets. The consumer may also sell some of her stock of assets and buy other assets. Choices are made to maximize expected utility, \( \pi u(c_1) + (1 - \pi) v(c_2) \). Given the nonconvexities due to the limited participation
restriction in period 0 and the fixed transactions costs in period 1, the agent's problem will ultimately involve discrete choice. That is, let $V_i$ denote the expected utility in period 0 to acquiring asset $i$. The consumer then solves

$$\max(V_1, V_2),$$

(1)

where

$$V_1 = \pi u(q_1 x_0) + (1 - \pi) \max(V_{11}, V_{12})$$

(2)

and

$$V_2 = \pi u[\max(0, q_2 x_0 - \alpha_1)] + (1 - \pi) \max(V_{21}, V_{22}, V_{23}),$$

(3)

with

$$V_{11} = v[\beta_2(q_1 x_0 + x_1 - \alpha_2)/q_2],$$

(4)

$$V_{12} = v(\beta_1 x_0 + \beta_1 x_1/q_1),$$

(5)

$$V_{21} = v(\beta_2 x_0 + \beta_1 x_1/q_1),$$

(6)

$$V_{22} = v[\beta_1(q_2 x_0 + x_1 - \alpha_1)/q_1],$$

(7)

$$V_{23} = v[\beta_2 x_0 + \beta_2(x_1 - \alpha_2)/q_2].$$

(8)

For the following, refer to fig. 1. Suppose an agent acquires the liquid asset in period 0. Then, if she is an early consumer, with probability $\pi$, she consumes $q_1 x_0$ in period 1 [eq. (2)]. If she is a late consumer, with probability $1 - \pi$, she either incurs the transactions cost, $\alpha_2$, and sells her stock of asset 1 and her consumption endowment to buy the illiquid asset, consuming $v^{-1}(V_{11})$ in period 2 [eq. (4)], or she sells the endowment for the liquid asset and retains the stock of the liquid asset that was acquired in period 0, consuming $v^{-1}(V_{12})$ in period 2 [eq. (5)]. Suppose the agent acquires the illiquid asset in period 0. If she is an early consumer, she either sells her stock of assets, incurring the cost $\alpha_1$, or consumes zero [eq. (3)]. If she is a late consumer, there are three possible actions that could be taken in period 1. First, she could hold the stock of the illiquid asset acquired in period 0 until period 2, and exchange the period 1 endowment for the liquid asset, consuming $v^{-1}(V_{21})$ in period 2 [eq. (6)]. Second, the stock of asset 2 and the period 1 endowment could be exchanged for the liquid asset, with a consumption level of $v^{-1}(V_{22})$ in period 2 [eq. (7)]. Third, the stock of the illiquid asset from period 0 could be held until the final period and the period
endowment exchanged for the illiquid asset, in which case \( v^{-1}(V_{23}) \) is consumed in period 2 [eq. (8)].

From (1)-(3), there are eight possible choices an agent could make, where a single choice defines the asset acquired in period 0 and actions taken in period 1 contingent on type (early or late consumer). Let \( U_j(q_1, q_2) \) denote the expected utility the agent achieves by taking choice \( j \), given prices, where \( j = 1, 2, \ldots, 8 \). From (1)-(3), these expected utilities are:

\[
U_1(q_1, q_2) = \pi u(q_1 x_0) + (1 - \pi) V_{11},
\]
\[
U_2(q_1, q_2) = \pi u(q_1 x_0) + (1 - \pi) V_{12},
\]
\[
U_3(q_1, q_2) = \pi u(0) + (1 - \pi) V_{21},
\]
\[
U_4(q_1, q_2) = \pi u(0) + (1 - \pi) V_{22},
\]
\[
U_5(q_1, q_2) = \pi u(0) + (1 - \pi) V_{23},
\]
\[
U_6(q_1, q_2) = \pi u(q_2 x_0 - \alpha_1) + (1 - \pi) V_{21},
\]
\[
U_7(q_1, q_2) = \pi u(q_2 x_0 - \alpha_1) + (1 - \pi) V_{22},
\]
\[
U_8(q_1, q_2) = \pi u(q_2 x_0 - \alpha_1) + (1 - \pi) V_{23}.
\]

Note that all choices may not be feasible given prices \( q_1 \) and \( q_2 \). For a particular choice \( j \) to be feasible, consumption must be nonnegative in each period. For example, choice 6 is not feasible if \( q_2 x_0 - \alpha_1 < 0 \) (consumption is negative in period 1 if the agent consumes early), and choice 8 is not feasible if \( x_1 - \alpha_2 < 0 \) (if the agent consumes late, the period 1 endowment is not large enough to cover the fixed transactions cost, i.e., consumption is negative in period 1). Let \( S_f \) denote the set of feasible choices, where \( S_f \in \{1, 2, 3, \ldots, 8\} \).

4. Competitive equilibrium

Given the nonconvexities in each agent's problem, we need to allow for the possibility that there may not be a unique choice which maximizes expected

\footnote{It may appear that choice has been left out in (1)-(8) above. That is, suppose an agent produces the liquid asset in period 0 and, if a late consumer, sells her endowment for the illiquid asset in period 1. Then, utility if a late consumer is \( V_{13} = v[\beta_1 x_0 + \beta_2(x_1 - \alpha_2)/q_2] \). However, note that this choice is always dominated. If \( \beta_1/q_1 \geq \beta_2/q_2 \), then \( V_{12} > V_{13} \), and if \( \beta_1/q_1 < \beta_2/q_2 \), then \( V_{13} > V_{11} \).}
utility given equilibrium prices, in which case different agents may make different choices. Let $p_j$ denote the fraction of agents who make choice $j$.

**Definition.** A competitive equilibrium is given by $\hat{q}_i, \hat{p}_j, i = 1, 2, j = 1, 2, \ldots, 8$, such that with $q_i = \hat{q}_i, p_j = \hat{p}_j, i = 1, 2, j = 1, 2, \ldots, 8$,

$$q_1(p_1 + \pi p_2)x_0 = (1 - \pi)[(p_2 + p_3 + p_6)x_1 + (p_4 + p_7)(q_2x_0 + x_1 - \alpha_1)]$$

[market clearing for the liquid asset in period 1],

$$q_2[\pi(p_6 + p_7 + p_8) + (1 - \pi)(p_4 + p_7)]x_0 = (1 - \pi)[p_1(q_1x_0 + x_1 - \alpha_2) + (p_5 + p_8)(x_1 - \alpha_2)]$$

[market clearing for the illiquid asset in period 1],

$$\sum_j p_j = 1,$$

$$q_i \geq 0, \quad 0 \leq p_j \leq 1, \quad i = 1, 2, \quad j = 1, 2, \ldots, 8,$$

if $p_j > 0$ then $U_j(q_1, q_2) \geq U_k(q_1, q_2)$ for all $k \in S_f$

[equilibrium choices are weakly preferred to all feasible choices].

In the period 1 market-clearing conditions for the two assets, (17) and (18), the left side of each equation represents the quantity of consumption goods received in period 1 by sellers of the asset, and the right side the quantity of consumption goods, net of transactions costs, exchanged by buyers of the asset. For example, in eq. (17), agents making choice 1 (fraction $p_1$ of the population) acquire the liquid asset in period 0 and then sell it under all contingencies in period 1. If an agent makes choice 2, the liquid asset is acquired, but sold only if the agent consumes early (with probability $\pi$). Thus, on the left side of (17), $p_1 + \pi p_2$ is the fraction of the population selling the liquid asset, and each seller receives $q_1x_0$ units of the consumption good. Agents buying the liquid asset are all late consumers. On the right side of (17), the fraction of the population exchanging their endowment for the liquid asset is $(1 - \pi)(p_2 + p_3 + p_6)$, and each of these agents has an endowment of $x_1$ units of the consumption good. Some agents [fraction $(1 - \pi)(p_4 + p_7)$ of the population] sell their holdings of the illiquid asset and their endowment (each agent exchanges $q_2x_0 + x_1 - \alpha_1$ units of the consumption good, net of transactions costs) for the liquid asset.
Eq. (19) states that the fractions of agents making each choice sum to unity, (20) requires nonnegativity of prices and that population fractions fall between zero and one, and (21) states that, if a given choice is made by a positive fraction of agents in equilibrium, then that choice must be weakly preferred to all other feasible choices.

4.1. Candidate equilibria

As a first step in determining competitive equilibria, it is useful to eliminate candidate equilibria which are clearly inconsistent with (17)–(20). Let \( S = \{ j : p_j > 0 \} \) denote the set of choices made by a positive fraction of agents in equilibrium, where \( S \subset S_r \). The following lemma is a useful first step.

**Lemma 1.** In a competitive equilibrium, \( q_1 > 0 \).

**Proof.** Suppose \( q_1 = 0 \). Then, from (17), we have \( p_2 = p_3 = p_6 = 0 \). If \( q_2 x_0 + x_1 - \alpha_1 > 0 \), then, from (17), \( p_4 = p_7 = 0 \). If \( q_2 x_0 + x_1 - \alpha_1 < 0 \), then choices 4 and 7 are not feasible, and \( p_4 = p_7 = 0 \). In the case where \( q_2 x_0 + x_1 - \alpha_1 = 0 \), we have \( q_2 x_0 - \alpha_1 < 0 \), and choice 7 is then not feasible, so \( p_7 = 0 \). Also, in this case \( U_2(q_1, q_2) < U_4(q_1, q_2) \), which implies, from (21), that \( p_4 = 0 \). Then, from (19), at least one of \( p_1, p_5, \) or \( p_8 \) must be positive. If \( x_1 < \alpha_2 \), then choice \( j \) is not feasible for \( j = 1, 5 \), or 8, a contradiction. If \( x_1 = \alpha_2 \), then \( q_2 = 0 \), from (18). Choice 8 is then not feasible, so \( p_8 = 0 \). From (9), (10), (11), and (13), (21) does not hold for \( j = 5 \) and \( k = 2 \), and for \( j = 1 \) and \( k = 2 \), a contradiction. If \( x_1 > \alpha_2 \), then from (18), \( q_2 > 0 \), and (21) does not hold for \( j = 1 \) and \( k = 2 \), for \( j = 5 \) and \( k = 3 \), and for \( j = 8 \) and \( k = 7 \). Q.E.D.

Essentially, the proof of Lemma 1 shows that, if the price of the liquid asset were zero in period 1, this would imply an infinite one-period rate of return to holding the liquid asset from period 1 to period 2. Optimizing choices would then imply excess demand for the liquid asset in period 1.

**Proposition 1.** \( 3 \notin S, 4 \notin S, 5 \notin S \).

**Proof.** Suppose that \( 3 \in S \), so that \( p_3 > 0 \). Given Lemma 1, (21) does not hold for \( j = 3 \) and \( k = 2 \), a contradiction. Similar arguments apply if \( 4 \in S \) or \( 5 \in S \). Q.E.D.

The logic behind Proposition 1 is the following. Choices 3, 4, and 5 imply zero consumption for early consumers. But, since the price of the liquid asset is positive in equilibrium and \( u'(0) = \infty \), choices 3, 4, and 5 are always dominated by choice 2.
Proposition 2. $7 \notin S$.

Proof. Suppose that $p_7 > 0$. Then (21) implies that $U_6(q_1, q_2) \leq U_7(q_1, q_2)$. Therefore, $\beta_1/q_1 > \beta_2/q_2$. Therefore, $U_2(q_1, q_2) > U_1(q_1, q_2)$ and $U_6(q_1, q_2) > U_8(q_1, q_2)$, which implies, from (21), that $p_1 = p_8 = 0$. Since $p_7 > 0$ implies $q_2 > 0$ (for feasibility), and given Lemma 1, (18) does not hold, a contradiction. Q.E.D.

From Propositions 1 and 2, agents will never choose, in equilibrium, to sell the illiquid asset to buy the liquid asset, if they are late consumers. If an agent acquires the illiquid asset in period 0 and is a late consumer, she will hold the illiquid asset until the final period (that is, choices 6 and 8 are the only possibilities for agents acquiring the illiquid asset in period 0).

Proposition 3. $S \neq \{1\}, S \neq \{6\}, S \neq \{1, 8\}, S \neq \{6, 8\}, S \neq \{1, 2\}, S \neq \{2, 6\}$.

Proof. If $S = \{1\}$, or $S = \{6\}$, or $S = \{1, 8\}$, or $S = \{6, 8\}$, then (17) does not hold given Lemma 1, a contradiction. If $S = \{1, 2\}$ or $S = \{2, 6\}$, then feasibility implies that $q_2 > 0$, but then (18) does not hold, a contradiction. Q.E.D.

Proposition 3 deals with cases where candidate equilibrium choices imply that there are sellers in an asset market and no buyers, or vice versa, thus violating market clearing.

Proposition 4. $S \neq \{2, 8\}, S \neq \{1, 2, 8\}, S \neq \{2, 6, 8\}, S \neq \{1, 2, 6, 8\}$.

Proof. Suppose that $S = \{2, 8\}$. Then (21) holds for $j = 8$ and $k = 6$ and for $j = 2$ and $k = 1$, or

$$ (\beta_2/q_2 - \beta_1/q_1) x_1 \geq \beta_2 \alpha_2/q_2 $$

and

$$ (\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) \leq \beta_2 \alpha_2/q_2. $$

But (22) implies that $\beta_2/q_2 - \beta_1/q_1 > 0$. Therefore, $(\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) > \beta_2 \alpha_2/q_2$, a contradiction. Now, suppose that $S = \{1, 2, 8\}$. Given (21) and (23), the following equality must hold:

$$ (\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) = \beta_2 \alpha_2/q_2. $$
But (22) implies that \((\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) > \beta_2 x_2/q_2\), a contradiction.

Next, suppose that \(S = \{2, 6, 8\}\). Then (23) must hold, in addition to
\[
(\beta_2/q_2 - \beta_1/q_1)x_1 = \beta_2 x_2/q_2 .
\] (25)

But (25) implies that \((\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) > \beta_2 x_2/q_2\), a contradiction.

Last suppose that \(S = \{1, 2, 6, 8\}\). Then (22) and (23) must both hold with equality. Therefore \(\beta_2/q_2 - \beta_1/q_1 > 0\), but then Lemma 1 implies that (22) and (23) cannot both hold with equality, a contradiction. Q.E.D.

Proposition 4 deals with some cases which violate optimality.

From Propositions 1–4, we are left with five candidate equilibria. That is, \(S = \{2\}\), \(S = \{8\}\), \(S = \{1, 6\}\), \(S = \{1, 2, 6\}\), and \(S = \{1, 6, 8\}\). We will denote each case by the set \(S\) that corresponds to it. Note that equilibrium choices are either 1, 2, 6, or 8. The fraction of agents acquiring the liquid asset in period 0 is \(p_1 + p_2\), and the fraction acquiring the illiquid asset is \(p_6 + p_8\). Equilibrium \{2\} has all agents acquiring and trading the liquid asset, in equilibrium \{8\} all agents acquire and trade the illiquid asset, and in equilibria \{1, 6\}, \{1, 2, 6\}, and \{1, 6, 8\} both assets are acquired and traded.

4.2. Existence and uniqueness of equilibrium

In the appendix, we show that there can exist at most one of each of the five candidate equilibria. Also in the appendix, necessary and sufficient conditions are established for the existence of each type of equilibrium. Next, we will show that an equilibrium always exists, and that it is unique. That is, at least one type of equilibrium exists, and no two types of equilibria coexist.

Define \(W(S)\) to be the subset of the parameter space for which an equilibrium of type \(S\) exists, \(S = \{2\}, \{8\}, \{1, 6\}, \{1, 2, 6\}, \{1, 6, 8\}\). From the series of Propositions in the appendix, we get the following:

\[
W(\{2\}) = \{\phi: F^1(\phi) \geq 0\} , \quad (26)
\]

\[
W(\{8\}) = \{\phi: F^4(\phi) \leq 0\} , \quad (27)
\]

\[
W(\{1, 6\}) = \{\phi: F^2(\phi) \leq 0, F^3(\phi) \geq 0\} , \quad (28)
\]

\[
W(\{1, 2, 6\}) = \{\phi: F^1(\phi) \leq 0, F^2(\phi) > 0\} , \quad (29)
\]

\[
W(\{1, 6, 8\}) = \{\phi: F^3(\phi) < 0, F^4(\phi) > 0\} . \quad (30)
\]
Here, $\phi \equiv (\pi, x_0, x_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is the parameter vector, and the functions $F^i(\phi), i = 1, 2, 3, 4$, are defined as follows:

\[
F^1(\phi) \equiv \pi u(1 - \pi) x_1/\pi + (1 - \pi) u(\beta_1 x_0/[1 - \pi]) \\
- \pi u(\beta_2 [x_1 - \pi x_0]/[1 - \pi] - \alpha_1) \\
- (1 - \pi) v(\beta_2 x_0 + \beta_1 x_0/\pi/[1 - \pi] ), \tag{31}
\]

\[
F^2(\phi) \equiv \pi u(\beta_2 [x_1 - \pi x_0]/[1 - \pi] - \alpha_1) \\
- \pi u(\beta_2 q_1 x_0 + x_1 - \alpha_2)/\beta_1 [q_1 x_0 + x_1] - \alpha_1) \\
- (1 - \pi) v(\beta_2 x_0 + \beta_1 x_1/q_1), \tag{32}
\]

\[
F^3(\phi) \equiv \pi u(\beta_2 [x_1 - \pi x_0]/[1 - \pi] - \alpha_1) \\
- \pi u(\beta_2 q_1 x_0 + x_1 - \alpha_2)/\beta_1 [q_1 x_0 + x_1] - \alpha_1) \\
- (1 - \pi) v(\beta_2 x_0 + \beta_1 x_1/q_1), \tag{33}
\]

\[
F^4(\phi) \equiv \pi u(\beta_2 [x_1 - \pi x_0]/\beta_2) \\
+ (1 - \pi) v(\beta_1 x_0 x_1/[x_1 - \alpha_2] + \beta_2 x_0/\beta_2) \\
- \pi u([1 - \pi] [x_1 - \alpha_2]/\pi - \alpha_1) - (1 - \pi) v(\beta_2 x_0/[1 - \pi] ) . \tag{34}
\]

From (32) and (33), $q_1$ and $\tilde{q}_1$ are defined, respectively, in (A.29) and (A.42) in the appendix. It is straightforward to show, from results in the appendix [see also Williamson (1991)], that

\[
F^1(\phi) < F^2(\phi) < F^3(\phi) < F^4(\phi) . \tag{35}
\]

From (26)-(30) and (35), the sets $W(S)$, $S = \{2\}, \{8\}, \{1, 6\}, \{1, 2, 6\}, \{1, 6, 8\}$, are disjoint, and $\bigcup_s W(S)$ is the parameter space. Therefore, an equilibrium always exists, and it is unique, given the results in the appendix.

It is straightforward to show that $W(S)$ is of positive measure for each $S$, so that there exists parameters for which an equilibrium of each type exists. First, note that $F^1(\phi) > 0$ for $\beta_1 = \beta_2$, which, from (26), implies that $W(2)$ is of positive measure, by continuity. Next, note that $F^4(\phi) = 0$ for $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2$. Now, $\partial F^4/\partial \beta_2 < 0$ when $\beta_1 = \beta_2$ so that, given (27) and again by
continuity, there exists a subset of the parameter space of positive measure where $F^4(\phi) < 0$, i.e., $W(\{8\})$ is of positive measure. Therefore, from (26)–(30) and (35), $W(S)$ is of positive measure for each $S$.

The parameter space can thus be divided into three regions of interest. First, there is a region where only the liquid asset is traded (equilibrium $\{2\}$); second, there is a region where only the illiquid asset is traded (equilibrium $\{8\}$); third, there exists a ‘diversification region’ where both assets are acquired and traded in equilibrium (either $\{1, 6\}$, $\{1, 2, 6\}$, or $\{1, 6, 8\}$). The locus $F^1(\phi) = 0$ separates the region in which the liquid asset is traded from the region where it is not. We have $\partial F^4 / \partial \alpha_1 > 0$, $\partial F^4 / \partial \alpha_2 > 0$, and $\partial F^4 / \partial \beta_2 |_{\delta = 0} < 0$. Therefore, the illiquid asset will be ‘more likely’ and the liquid asset ‘less likely’ to be acquired and traded, as transactions costs decrease and the return on the illiquid asset increases, as intuition tells us should be the case.

In the diversification region, some agents (those making choice 1) acquire the liquid asset in period 0 and purchase the illiquid asset in period 1 if they are late consumers. However, some other agents (those making choice 6) acquire the illiquid asset in period 0 and purchase liquid assets in period 1 if they are late consumers. The fact that some agents make choice 1 and some make choice 6 in equilibrium may appear inconsistent. Agents can be indifferent in equilibrium between choices 1 and 6 as, in such an equilibrium, acquiring the liquid asset implies a lower two-period rate of return, but a higher one-period rate of return, net of transactions costs. That is, the liquid asset provides a liquidity service, but an inferior long-run rate of return. In period 0, agents can then be indifferent between the two assets. For an agent who learns she is a late consumer in period 1, if the agent is holding the illiquid asset she will prefer to hold it until the final period for its one-period return. If the agent were holding the liquid asset, the size of the transaction she wishes to make in the asset market is larger than if she holds the illiquid asset. Thus, given the fixed transactions cost in the illiquid asset market, the agent holding the illiquid asset prefers to make her smaller transaction in the liquid asset market, while the agent holding the liquid asset makes a larger transaction on the illiquid asset market.

It is useful at this point to note the effects of relaxing the period 0 market participation restriction. The fact that agents cannot diversify between the two assets in period 0 is not a binding constraint in some regions of the parameter space, where the fixed transactions costs will imply that, even if agents could diversify in period 0, they strictly prefer not to at equilibrium prices. However, the constraint on diversification will certainly bind for some parameter values. In particular, holding other parameter values constant, suppose that the return on the illiquid asset, $\beta_2$, becomes very large, while the transactions costs, $\alpha_1$ and $\alpha_2$, also become large. Then, given the choice, each agent would choose a diversified portfolio in period 0 which includes the liquid and illiquid assets. Given this behavior, the liquid asset would be the only asset exchanged in period 1, and the illiquid asset would be held only for period 2 consumption. The period 0 port-
folio restriction adds to the model's tractability, but should not affect the results in an important way.

The model is capable of making predictions concerning asset prices, but it also determines the volume of trade in asset markets. From eqs. (17) and (18), the quantity of the liquid asset traded in period 1 is \((p_1 + \pi p_2)x_0\) and the quantity of the illiquid asset traded is \(\pi(p_6 + p_8)x_0\). Thus, if we measure aggregate trading volume by giving the quantity of each asset traded equal weight, then total volume traded is equal to \([(1 - \pi)p_1 + \pi]x_0\). Thus, trading volume is increasing in \(p_1\). In the regions of the parameter space where \(p_1 = 0\), only one asset is acquired and traded, while \(p_1 > 0\) in all region of the parameter space where both assets are held. Therefore, when agents diversify in the aggregate, this tends to produce more asset trading.

5. The optimal provision of liquidity

The objective of this section is to derive a welfare result showing that liquidity tends to be underprovided in a competitive equilibrium, relative to a social optimum. In evaluating the optimality of the competitive equilibrium, we follow the spirit of work on incomplete markets [e.g., Geanakoplos and Polemarchakis (1986)]. Here, we allow the social planner to determine the fractions of agents holding liquid and illiquid assets in period 0. We let \(\gamma\) denote the fraction of agents initially holding the liquid asset, i.e., \(\gamma = p_1 + p_2\). The planner must respect the restriction on diversification, and agents are free to trade assets on competitive markets in period 1. Thus, the set of allocations that the planner can achieve are competitive equilibria, indexed by \(\gamma \in [0, 1]\). That is, a feasible allocation for the planner is a competitive equilibrium, satisfying (17)-(20), but imposing the constraint

\[
p_1 + p_2 = \gamma, \quad \gamma \in [0, 1]. \tag{36}
\]

In addition, (21) is replaced by

\[
\text{if } p_j > 0, \quad \gamma > 0, \quad \text{then} \quad U_j(q_1, q_2) \geq U_k(q_1, q_2),
\]

\[
j, k = 1, 2, \tag{37}
\]

\[
\text{if } p_j > 0, \quad \gamma < 1, \quad \text{then} \quad U_j(q_1, q_2) \geq U_k(q_1, q_2),
\]

\[
j, k = 3, 4, \ldots, 8. \tag{38}
\]

From (37) and (38) agents cannot choose their actions in period 0, but they make optimal choices in period 1.
We define an allocation to be constrained Pareto optimal if there is no allocation which is feasible for the planner and makes some agent better off without making any agent worse off. Clearly, there exist parameter values for which the competitive equilibrium will be constrained Pareto optimal. First, if the return on the illiquid asset, $\beta_2$, is sufficiently large and transactions costs sufficiently small, then equilibrium $\{8\}$, where agents hold and trade the illiquid asset only, will be the competitive equilibrium, from (27) and (34). Provided that $\beta_2$ is sufficiently large and $\alpha_1$ and $\alpha_2$ sufficiently small, this competitive equilibrium could not be improved upon by forcing some agents to hold the liquid asset, and is therefore constrained Pareto optimal. Second, transactions costs can always be made sufficiently large so that, from (26) and (31), the competitive equilibrium is $\{2\}$, where only the liquid asset is held and traded. If transactions costs are large enough that they would absorb an early consumer's resources in period 1, then forcing agents to hold the illiquid asset forces them to consume zero if they are early consumers. In this case the competitive equilibrium would be constrained Pareto optimal. Therefore, in regions of the parameter space where agents hold only the liquid or illiquid asset in equilibrium, we are less likely to find competitive equilibria which are constrained suboptimal. If our goal is to seek out the existence of suboptimal equilibria, a natural region of the parameter space in which to look is near or in the region where both assets are held and traded.

Now, we will show that there are conditions under which suboptimal equilibria, with too little liquidity provision, exist near the locus $F^I(\phi)$. This locus defines a boundary between the regions where equilibrium $\{2\}$ (agents hold only the liquid asset) exists and where equilibrium $\{1, 2, 6\}$ (agents hold both assets) exists. Assume that

$$u'(x) \geq v'(x) \quad \text{for all } x > 0,$$

and

$$\pi \geq x_1/(\beta_1 x_0 + x_1).$$

(39)

(40)

In the region defined by $F^I(\phi) \geq 0$, we have equilibrium $\{2\}$, from (26). The adjacent region of the parameter space is $W(\{1, 2, 6\})$, from (29), so that both regions have a positive fraction of agents making the choice 2, and expected utility is equal to $U_2(q_1, q_2)$. From the appendix, $q_1$ is a continuous function of the parameters at the locus $F^I(\phi) = 0$, which separates the regions $W(\{2\})$ and $W(\{1, 2, 6\})$. Differentiating expected utility with respect to $q_1$ and evaluating the derivative along $F^I(\phi) = 0$, using (10) and (17) [which implies $q_1 = (1 - \pi)x_1/\pi x_0$ along the locus $F^I(\phi) = 0$], we get

$$(d/dq_1)|_{q_1 = (1 - \pi)x_1/\pi x_0}[\pi u(q_1 x_0) + (1 - \pi)v(\beta_1 x_0 + \beta_1 x_1/q_1)]$$
\[
\pi x_0/(1 - \pi) x_1 \{ (1 - \pi) u'[(1 - \pi) x_1/\pi] - \pi x_0 \beta_1 v'(\beta_1 x_0/(1 - \pi)) \\
> \pi x_0/(1 - \pi) x_1 \{ (1 - \pi) u'[(1 - \pi) x_1/\pi] - \pi x_0 \beta_1 v'(\beta_1 x_0) \}
\]
\[
= \pi^2 x_0/(1 - \pi) x_1 \{ (1 - \pi) u'[(1 - \pi) x_1/\pi] - x_0 \beta_1 v'(\beta_1 x_0) \}
\]
\[
\geq \pi^2 \beta_1 x_0^2/(1 - \pi) x_1 [u'(\beta_1 x_0) - v'(\beta_1 x_0)]
\]
\[
\geq 0.
\]

Here, the second inequality follows from \(-cu''(c)/u'(c) \geq 1\) and (40), while the third inequality is implied by (39). Now, from the appendix, \(q_1 < \bar{q}_1\) in equilibrium \(\{1, 2, 6\}\), where \(\bar{q}_1\) is the price of the liquid asset in equilibrium \(\{2\}\). Therefore, from the above, expected utility falls, given (39) and (40), when a parameter change (such as an increase in \(\beta_2\) or a reduction in transactions costs which does not affect the expected utility of an agent making choice 2 except through \(q_1\)) causes the illiquid asset to be adopted. From the planner's point of view, equilibrium \(\{2\}\) is still feasible in region \(W(\{1, 2, 6\})\) by simply setting \(\gamma = 1\). Therefore, equilibrium \(\{1, 2, 6\}\) is suboptimal, given (30) and (40), in a neighborhood of the boundary \(F^1(\phi) = 0\).

The above shows that, if liquidity matters enough, then expected utility falls when a parameter change causes the illiquid asset to be adopted. Liquidity matters sufficiently if the marginal utility of early consumption is sufficiently large relative to the marginal utility of late consumption [(39) holds] and if the probability of early consumption is sufficiently large [(40) holds]. Under these circumstances, agents would be better off if they continued to hold the liquid asset rather than having some trade in illiquid assets, and the competitive equilibrium is suboptimal.

A key to the reasons for the underprovision of liquidity is the behavior of asset prices in the neighborhood of the locus \(F^1(\phi) = 0\). As indicated above, the price of the liquid asset is lower in equilibrium \(\{1, 2, 6\}\) than in equilibrium \(\{2\}\). To determine the behavior of the illiquid asset's price, note that the expected utility of an agent making choice 6, \(U_6(q_1, q_2)\), will increase (everything else held constant) when \(q_1\) falls, from (4) and (9). Parameter changes which result in the adoption of the illiquid asset also will tend to increase \(U_6(q_1, q_2)\). However, if (39) and (40) hold, then \(U_2(q_1, q_2)\) [equal to \(U_6(q_1, q_2)\)] must fall at the boundary \(F^1(\phi) = 0\). Since, from (4) and (9), \(U_6(q_1, q_2)\) is increasing in \(q_2\), the price of the illiquid asset then will tend to fall as the illiquid asset is adopted. Why do both asset prices tend to fall under these circumstances? This occurs because more trade in the illiquid asset absorbs more of the period 1 endowment in the form of transactions costs, thus reducing the demand for assets in period 1. This movement in asset prices constitutes an externality, due to the fact that there are missing contingent claims markets. That is, decreases in both asset
prices imply that the two assets provide less insurance against the need for liquidity. In adopting the illiquid asset, agents do not take into account the negative externality that results because of the effects of their collective actions on asset prices.

Because the effect of the adoption of the illiquid asset is to reduce asset prices, thus tending to produce a negative externality, it is unlikely that we could find circumstances under which liquidity would be overprovided. For example, consider the boundary between the regions of the parameter space where equilibrium \{8\} exists and where equilibrium \{1, 6, 8\} exists, defined by \(F^4(\phi) = 0\), from (27) and (30). Here, only the illiquid asset is held and traded, but if a small parameter change causes the liquid asset to be adopted, then asset prices will tend to rise, as a smaller quantity of the period 1 endowment is absorbed by transactions costs. But then both assets become more useful to agents in self-insuring individual risk, and welfare will tend to rise.

Our model is similar to models of market participation with coordination failures, for example, Chatterjee (1988), Pagano (1989a, b), and Allen and Gale (1991). In these coordination failure models there is market incompleteness and some form of externality, as is the case here. However, there are at least three important differences between the model constructed here and related coordination failure models. First, in these other models there is a strategic complementarity which produces multiple Pareto-ranked equilibria. Here, there is no strategic complementarity, in that when agents hold more of the illiquid asset, asset prices do not move in such a way as to encourage more participation in the illiquid asset market. As a result, the competitive equilibrium is unique. Second, the externalities in the coordination failure models tend to affect market participation in a direction which is opposite from what we get here. For example, in Pagano (1989a), there exist multiple Pareto-ranked equilibria, and the inferior equilibrium has a relatively small quantity of the risky asset held. If we think of the risky asset in Pagano’s model as playing the same role as the illiquid asset in our model, then Pagano’s model tends to exhibit underparticipation in this asset market, while our model tends to produce overparticipation. Third, the role of uninsured idiosyncratic risk is different in the coordination failure models and here (though Allen and Gale’s model also has preference shocks). In the coordination failure models, the idiosyncratic risk faced by an agent depends in general on actions taken by that agent, while in our model idiosyncratic risk comes from exogenous random liquidity needs.

6. Examples

To give a better feel for how the model works, and for the possible equilibrium outcomes, some equilibria were computed numerically. For the numerical examples, we assume \(u(c) = \ln(c)\), \(v(c) = \ln(c)\), \(\beta_1 = x_0 = x_1 = 1\), and
$\alpha_1 = \alpha_2 = \alpha$. We thus constrain the two-period interest rate on the liquid asset to be zero, and the equilibria computed have $\beta_2 \in (1, 1.1]$, so that the two-period interest rate on the illiquid asset is at most 10%. We used three different values for the probability of early consumption, $\pi = 0.1$, $\pi = 0.5$, and $\pi = 0.9$. For each value of $\pi$, equilibria were computed over a two-dimensional grid in $(\alpha, \beta_2)$ space.

In figs. 2–4, for each value of $\pi$ the parameter space is subdivided into the regions where each type of equilibrium exists. In some circumstances, the region of the parameter space where a particular equilibrium exists is too small to distinguish in the figure. This is the case, for example, for equilibrium $\{1, 6, 8\}$ in fig. 2. The horizontal scales differ among the figures so that the regions of the parameter space can be distinguished as much as possible. In figs. 2–4, note that for a given value of $\beta_2$, equilibria with trade in the illiquid asset tend to be supported with higher transactions costs, the smaller $\pi$ is. The illiquid asset is traded in equilibria $\{8\}$, $\{1, 6, 8\}$, $\{1, 6\}$, and $\{1, 6, 8\}$. In fig. 2, trade in the illiquid asset is supported with $\beta_2 = 1.05$, $\alpha = 1.5$, and $\pi = 0.1$, i.e., there is trade in the illiquid asset with a cost of buying or selling this asset 1.5 times the endowment of late consumers in period 1. However, in fig. 4, with $\beta_2 = 1.05$, $\alpha = 0.006$, and $\pi = 0.9$, there is no trade in the illiquid asset. Also note, from figs. 2–4, that the larger $\pi$ is, the smaller is the range of transactions costs, for given $\beta_2$, over which both assets are acquired and traded (the region where equilibria $\{1, 6, 8\}$, $\{1, 6\}$, and $\{1, 2, 6\}$ exist).

Fig. 2. Equilibria, $\pi = 0.1$. 
Fig. 3. Equilibria, $\pi = 0.5$.

Fig. 4. Equilibria, $\pi = 0.9$. 
Figs. 5, 7, and 9 are three-dimensional plots of expected utility computed over the same grids as for figs. 2, 3, and 4, respectively. Figs. 6, 8, and 10 are cross-sections of figs. 5, 7, and 9, respectively, with $\beta_2 = 1.05$. In figs. 5 and 6, with $\pi = 0.1$, expected utility decreases monotonically for fixed $\beta_2$ as $\alpha$
increases. That is, higher transactions costs imply lower welfare. In contrast, with higher $\pi$ in figs. 7–10, expected utility can rise as $\alpha$ increases. Note that, for $\pi = 0.9$, from figs. 4 and 10, that with $\beta_2 = 1.05$ expected utility is lower through much of the region where equilibrium [8] exists (only the illiquid asset is traded)
than in the region where equilibrium \{2\} exists (only the liquid asset is traded). Figs. 8, 11, and 12 illustrate why expected utility can fall when transactions costs decrease. Regions where this happens are also regions where the fraction of agents acquiring the liquid asset (γ, determined in the market) and asset prices
Fig. 11. $\pi = 0.5, \beta_2 = 1.05$.

Fig. 12. $\pi = 0.5, \beta_2 = 1.05$. 
decrease dramatically. Thus, as transactions costs fall, participation in the market for the illiquid (liquid) asset rises (falls) in such a way that the prices of both assets fall. As discussed in the previous section, the decrease in asset prices makes agents worse off, as assets then provide less insurance against random liquidity needs. Note that the kinks in fig. 8 do not correspond exactly to the boundaries in fig. 3. This is due to the coarseness in the grid over which equilibria were computed (a 20 \times 20 grid).

In figs. 13–19, expected utility is computed for equilibria where a social planner constrains the fraction of agents holding each asset at the initial date. Here, \( \gamma \) is the fraction of agents holding the liquid asset, \( u_1 \) is the expected utility of those agents, and \( u_2 \) is the expected utility of agents holding the illiquid asset. In fig. 13, the competitive equilibrium is where \( u_1 \) and \( u_2 \) intersect near \( \gamma = 0.1 \), and this is constrained Pareto optimal. Similarly, in fig. 14, the competitive equilibrium is at \( \gamma = 1 \), and it is constrained Pareto optimal. No suboptimal equilibria were found for \( \pi = 0.1 \).

Now, consider fig. 15. Here, the competitive equilibrium is at \( \gamma = 0 \), but all agents are better off if \( \gamma = 1 \), so the competitive equilibrium is suboptimal. In fig. 16, the competitive equilibrium is located where \( u_1 \) intersects \( u_2 \) (near \( \gamma = 0.35 \)), and again this is suboptimal, being dominated by \( \gamma = 1 \). In fig. 17, the competitive equilibrium is \( \gamma = 1 \), which is constrained Pareto optimal. In figs. 15–17, there is in general too little liquidity in a competitive equilibrium and \( \gamma = 1 \), though a constrained Pareto optimum is not a unique constrained Pareto
Fig. 14. Constrained equilibria, $\pi = 0.1, \beta_2 = 1.05, x = 3$.

Fig. 15. Constrained equilibria, $\pi = 0.5, \beta_2 = 1.05, x = 0.025$. 
Fig. 16. Constrained equilibria, $\pi = 0.5$, $\beta_2 = 1.05$, $\tau = 0.04$.

Fig. 17. Constrained equilibria, $\pi = 0.5$, $\beta_2 = 1.05$, $\tau = 0.1$. 
Fig. 18. Constrained equilibria, \( \pi = 0.9, \beta_2 = 1.05, \alpha = 0.001 \).

Fig. 19. Constrained equilibria, \( \pi = 0.9, \beta_2 = 1.05, \alpha = 0.005 \).
optimum. Similarly, with $\pi = 0.9$ in fig. 18, $\gamma = 0$ is the competitive equilibrium, which is suboptimal as it is dominated by $\gamma = 1$, but there is more than one Pareto optimum. In figs. 19 and 20, however, $\gamma = 1$ is the unique constrained Pareto optimum. In fig. 19, $\gamma = 1$ dominates the competitive equilibrium, where $u_1$ and $u_2$ intersect (near $\gamma = 0.6$) and, in fig. 20, $\gamma = 1$ is the competitive equilibrium.

These examples illustrate a tendency for a higher probability of a need for liquidity (higher $\pi$) to produce suboptimality with greater likelihood. In none of the examples is it the case that too much liquidity is provided; in the examples equilibria are either constrained Pareto optimal or are dominated by feasible allocations with more liquidity and less participation in the illiquid asset market.

7. Conclusion

The model constructed here is one of multiple asset markets, which permits the volume of trade on each market to be determined endogenously. The possible outcomes involve either trade in both assets, trade in only the liquid asset, or trade in only the illiquid asset. Given the nonconvexities in agent's optimization problems due to fixed transactions costs and the constraint that agents cannot initially diversify their portfolios, different agents may participate
in different markets and make different portfolio decisions. This occurs in spite of the fact that agents are identical, ex ante.

In this model, there is a tendency for underprovision of the liquid asset. The model shares this property with some standard monetary models, for example models with cash-in-advance constraints or with money in the utility function. An important difference here, in contrast to these other models, is that there is not some set of assets which is defined at the outset as providing liquidity services. Here, any asset can potentially provide liquidity services (i.e., insure against random liquidity needs) but some assets perform better in this role than others. The model also shares some features, such as incomplete markets and a participation externality, with coordination failure models of market participation. However, while the coordination failure models contain strategic complementarities which generate multiple equilibria and a tendency for underprovision of liquid assets, the competitive equilibrium in our model is unique and there tends to be an overprovision of illiquid assets. The participation externality in our model is also quite different, in that more participation in the illiquid asset market absorbs more transactions costs, causing asset prices to fall in the trading period, which in turn implies that assets provide less insurance against random liquidity needs. Agents do not take account of the effect their participation decisions have on asset prices.

One possible extension of the model would be to introduce some aggregate uncertainty by making the fraction of early consumers random as of period 0. As a result, the prices of the liquid and illiquid assets would also be random. This could possibly create participation externalities of the type studied by Chatterjee (1988), Pagano (1989a), and Allen and Gale (1991), and might provide an interesting interaction with the externality already present here. Another extension could involve embedding the model in an overlapping generations framework in which the liquid asset is fiat currency used for intergenerational trade. Here, government monetary interventions affecting rates of return would change the degree of market participation [see Chatterjee and Coabe (1992)], and the equilibrium effects would depend critically on the initial degree of market participation.

Appendix

The appendix shows, through a series of propositions, that each of the five types of candidate competitive equilibria, from section 4.1, is unique within its class, if it exists. Further, we establish necessary and sufficient conditions for the existence of each type of candidate equilibrium.
Equilibrium \{2\}

Here, \( p_2 = 1 \) and \( p_j = 0 \) for \( j = 2, 3, \ldots, 8 \). From the market-clearing condition (17) we can then solve for the price of the liquid asset to get

\[
q_1 = (1 - \pi)x_1/\pi x_0.
\]  
(A.1)

Proposition A.1. Equilibrium \{2\} is unique (except for \( q_2 \), a price at which no assets are traded) within its class, if it exists, and it exists if and only if

\[
\pi u(\frac{[1 - \pi] x_1}{\pi}) + (1 - \pi) v(\beta_1 x_0/[1 - \pi])
\geq \pi u(\beta_2 [x_1 - \pi x_2]/\beta_1 \pi - \alpha_1)
+ (1 - \pi)v(\beta_2 x_0 + \beta_1 x_0 \pi/[1 - \pi]).
\]  
(A.2)

Proof. Given \( p_2 = 1 \) and \( p_j = 0 \) for all \( j \neq 2 \), and (A.1), (17)–(20) are satisfied by construction, except for \( q_2 \geq 0 \). For an equilibrium, we must find a price for the illiquid asset, \( q_2 \geq 0 \), such that (21) is satisfied for \( j = 2 \) and \( k \neq 2 \), given (A.1). With \( j = 2 \) and \( k = 1 \) in (21), and substituting using (A.1), we get

\[
\beta_1 x_0/(1 - \pi) \geq (\beta_2/q_2)(x_1/\pi - x_2).
\]  
(A.3)

Similarly, inequality (21) with \( j = 2 \) and \( k = 6 \) gives

\[
\pi u(\frac{[1 - \pi] x_1}{\pi}) + (1 - \pi) v(\beta_1 x_0/[1 - \pi])
\geq \pi u(q_2 x_0 - x_1) + (1 - \pi)v(\beta_2 x_0 + \beta_1 \pi x_0/[1 - \pi]).
\]  
(A.4)

Inequalities (A.3) and (A.4) imply lower and upper bounds, respectively, that \( q_2 \) must satisfy in equilibrium. If (A.3) holds, then

\[
U_2(q_1, q_2) \leq \pi u(\frac{[1 - \pi] x_1}{\pi}) + (1 - \pi)v(\beta_1 x_0/[1 - \pi])
- (1 - \pi)v(\beta_2 x_0 + \beta_1 x_0 \pi/[1 - \pi])
+ (1 - \pi)v(\beta_2 x_0 + \beta_1 x_0 \pi[x_1 - \pi x_2]/[1 - \pi])
+ (1 - \pi)[x_1 - \pi x_2]
\]  
(A.5)
Then, if (A.3) holds, we have \( U_8(q_1, q_2) < U_2(q_1, q_2) \). Next, we need to show that (21) holds for \( j = 2 \) and \( k = 7 \). If (A.3) holds with equality, then

\[
U_6(q_1, q_2) - U_7(q_1, q_2) \\
= (1 - \pi)v(\beta_2 x_0 + \beta_1 \pi x_0 / [(1 - \pi)x_1]) - (1 - \pi)v(\beta_2 (x_1 - \pi x_2)x_0/x_1 \\
+ \beta_1 \pi x_0/(1 - \pi) - \beta_1 \pi x_0 x_1 / [(1 - \pi)x_1]) > 0.
\]

In addition, (21) holds for \( j = 2 \) and \( k = 3, 4, 5 \), since \( u'(0) = \infty \). Therefore, equilibrium \{2\} exists if and only if there is a \( q_2 > 0 \) satisfying (A.3) and (A.4). Since (A.3) defines a lower bound on \( q_2 \) and (A.4) an upper bound on \( q_2 \), we can take (A.3) as an equality, solve for \( q_2 \), and substitute in (A.4). Then, equilibrium \{2\} exists if and only if (A.2) holds. If equilibrium \{2\} exists, it is essentially unique, which is trivial by construction and from (A.1). Q.E.D.

**Equilibrium \{8\}**

In equilibrium \{8\}, we have \( p_8 = 1 \) and \( p_j = 0 \) for \( j = 1, 2, \ldots, 7 \). Then, from (18), the equilibrium price of the illiquid asset is

\[
q_2 = (1 - \pi)(x_1 - \alpha_2)/\pi x_0. \tag{A.6}
\]

**Proposition A.2.** Equilibrium \{8\} is essentially unique (except for the price \( q_1 \), at which no trade takes place) within its class, if it exists, and it exists if and only if

\[
\pi u([1 - \pi][x_1 - \alpha_2]/\pi - \alpha_1) + (1 - \pi)v(\beta_2 x_0 / [1 - \pi]) \\
\geq \pi u(\beta_1 x_1 [1 - \pi]/\pi \beta_2) \\
+ (1 - \pi)v(\beta_1 x_0 x_1 / [x_1 - \alpha_2] + \beta_2 \pi x_0 / [1 - \pi]). \tag{A.7}
\]

**Proof.** Inequality (21) must be satisfied for \( j = 8 \) and \( k = 1 \), or using (A.6),

\[
\pi u(q_1 x_0) + (1 - \pi)v(\beta_2 x_0^2 \pi q_1 / [1 - \pi][x_1 - \alpha_2] + \beta_2 \pi x_0 / [1 - \pi]) \\
\leq \pi u([1 - \pi][x_1 - \alpha_2]/\pi - \alpha_1) + (1 - \pi)v(\beta_2 x_0 / [1 - \pi]). \tag{A.8}
\]
Similarly, (21) must hold for \( j = 8 \) and \( k = 6 \), or

\[
\beta_1 x_1 / q_1 \leq \beta_2 x_0 \pi / (1 - \pi) .
\] (A.9)

Inequalities (A.8) and (A.9) put upper and lower bounds, respectively, on \( q_1 \). From (10), if (A.8) holds and (A.9) holds with equality, then

\[
U_2(q_1, q_2) \\
\leq U_8(q_1, q_2) - (1 - \pi)\nu(\beta_1 x_0 x_1/[x_1 - x_2] + \beta_2 \pi x_0/[1 - \pi]) \\
+ (1 - \pi)\nu(\beta_1 x_0 + \beta_2 x_0 \pi/[1 - \pi]) \\
< U_8(q_1, q_2) ,
\]

so that (21) holds for \( j = 8 \) and \( k = 2 \). If (A.9) holds with equality, then, using (14) and (15),

\[
U_6(q_1, q_2) - U_7(q_1, q_2) \\
= \beta_2 x_0 - \beta_2 x_0 (x_1 - x_2)/x_1 + \alpha_1 \beta_2 \pi x_0/(1 - \pi) x_1 > 0 ,
\]

Thus, if (A.8) and (A.9) hold, then there exists a \( q_1 \) satisfying (A.8) and (A.9), which guarantees that (21) holds for \( j = 8 \) and \( k = 7 \). Also, (21) holds for \( j = 8 \) and \( k = 3, 4, 5 \), given (A.8). Therefore, in a manner similar to the proof of Proposition A.1, equilibrium \{8\} exists if and only if there exists a \( q_1 \geq 0 \) satisfying (A.8) and (A.9), that is, taking (A.9) with equality and substituting in (A.8), we get (A.7). By construction and from (A.8), it is trivial that equilibrium \{8\} is essentially unique if it exists. Q.E.D.

**Equilibrium \{1, 6\}**

Here, \( p_1 > 0, p_6 > 0 \), and \( p_j = 0 \) for \( j = 2, 3, 4, 5, 8 \). Then, from (17), we get

\[
q_1 = (1 - \pi)p_6 x_1/p_1 x_0 ,
\] (A.10)

and, from (18) and (A.10),

\[
q_2 = (1 - \pi)p_1 (x_1 - x_2)/\pi p_6 x_0 + (1 - \pi)^2 x_1/\pi x_0 .
\] (A.11)
Since $p_i > 0$ for $i = 1, 6$, then (21) must hold with equality for $j = 1$ and $k = 6$, or using (9), (14), (A.10), and (A.11),

$$
\pi u([1 - \pi]p_6 x_1/p_1) + (1 - \pi)v(\beta_2 \pi x_0 p_6/[1 - \pi]p_1) \\
= \pi u([1 - \pi]p_1 [x_1 - \alpha_2]/\pi p_6 + [1 - \pi]^2 x_1/\pi - \alpha_1) \\
+ (1 - \pi)v(\beta_2 x_0 + \beta_1 x_0 p_1/p_6 [1 - \pi]).
$$

(A.12)

From (19), we have

$$p_1 + p_6 = 1.
$$

(A.13)

The values of $p_1$, $p_6$, $q_1$, and $q_2$ which solve (A.10)–(A.13) constitute an equilibrium.

**Proposition A.3.** If equilibrium $\{1, 6\}$ exists, it is unique within its class. Further, equilibrium $\{1, 6\}$ exists if and only if

$$
\pi u([1 - \pi]\tilde{\theta} x_1) + (1 - \pi)v(\beta_1 x_0 + \beta_1 x_0/[1 - \pi]\tilde{\theta}) \\
\leq \pi u(\beta_2 [1 - \pi]\tilde{\theta} [x_1 - \alpha_2]/\beta_1 - \alpha_1) \\
+ (1 - \pi)v(\beta_2 x_0 + \beta_1 \pi x_0/[1 - \pi]\tilde{\theta})
$$

(A.14)

and

$$
\pi u([1 - \pi]x_1\tilde{\theta}) + (1 - \pi)v(\beta_1 x_0 x_1/[x_1 - \alpha_2] + \beta_1 x_0 \pi/[1 - \pi]\tilde{\theta}) \\
\geq \pi u(\beta_2 [x_1 - \alpha_2][1 - \pi]\tilde{\theta}/\beta_1 \pi - \alpha_1) \\
+ (1 - \pi)v(\beta_2 x_0 + \beta_1 \pi x_0/[1 - \pi]\tilde{\theta}),
$$

(A.15)

where

$$\tilde{\theta} \equiv \left\{ \beta_1 (1 - \pi) + [\beta_2^2 (1 - \pi)^2 + 4\beta_1 \beta_2 \pi]^{1/2} \right\}/2\beta_2 \pi
$$

(A.16)

and

$$\bar{\theta} \equiv \left\{ \beta_1 (1 - \pi)x_1 + [\beta_2^2 (1 - \pi)^2 x_1^2 + 4\pi \beta_1 \beta_2 x_1 - \alpha_2]^{1/2} \right\} \\
/2\pi \beta_2 (x_1 - \alpha_2).
$$

(A.17)
Proof. Inequality (21) must hold for $j = 1$ and $k = 2$, that is, using (9), (10), (A.15), and (A.16),

$$
\beta_2 \pi (p_6/p_1)(1 - \alpha) \geq \beta_1 + \beta_1/(1 - \alpha)(p_6/p_1). \tag{A.18}
$$

Also, (21) must hold for $j = 6$ and $k = 8$. Using (14), (16), (A.10), and (A.11), we get

$$
\beta_1(x_1 - \alpha_2)/\pi(p_6/p_1)^2 + \beta_1(1 - \alpha)x_1/\pi(p_6/p_1) \geq \beta_2(x_1 - \alpha_2). \tag{A.19}
$$

Now, if (21) holds for $j = 1$ and $k = 2$, from (9) and (10), we have $\beta_2/q_2 - \beta_1/q_1 > 0$, which implies that, from (14) and (15), (21) holds for $j = 6$ and $k = 8$. Inequality (21) holds for $j = 1, 6$ and $k = 3, 4, 5$, as (A.10) implies that $q_1 > 0$, so that choice 1 dominates choices 3, 4, and 5, and choice 1 is in turn dominated by choice 2.

Let $\theta = p_6/p_1$. Then, since the left side of (A.12) is increasing in $\theta$ and the right side is decreasing in $\theta$, and given the restrictions on $u(\cdot)$ and $v(\cdot)$, eq. (A.12) yields a unique solution for $\theta$, which in turn gives unique solutions for $q_1, q_2, p_1, p_6$ from (A.10), (A.11), and (A.12), and the solutions satisfy (20). We then have a unique equilibrium, of type $\{1, 6\}$ if the solution to (A.12), denoted by $\theta^*$, satisfies (A.18) and (A.19). Using (A.18) and (A.19), we then get $\theta^* \geq \theta$, and $\theta^* \leq \theta$. Then, from (A.12), a unique equilibrium of type $\{1, 6\}$ exists if and only if (A.14) and (A.15) hold. Q.E.D.

Equilibrium $\{1, 2, 6\}$

With equilibrium $\{1, 2, 6\}$, $p_1 > 0$, $p_2 > 0$, $p_6 > 0$, and $p_j = 0$ for $j = 3, 4, 5, 7, 8$. From (17), we get

$$
q_1 = (1 - \pi)(p_2 + p_6)x_1/(p_1 + \pi p_2)x_0, \tag{A.20}
$$

and, from (18) and (A.20),

$$
q_2 = (1 - \pi)p_1(x_1 - \alpha_2)/\pi p_6 x_0

+ (1 - \pi)^2 p_1(p_2 + p_6)x_1/\pi p_6(p_1 + \pi p_2)x_0. \tag{A.21}
$$

Here, (21) must hold with equality for $j = 1$ and $k = 2$, that is,

$$
q_2 = \beta_2(q_1 x_0 + x_1 - \alpha_2)/(\beta_1 x_0 + \beta_1 x_1/q_1). \tag{A.22}
$$
Also, (21) must hold for \( j = 2 \) and \( k = 6 \), or, using (A.22) to substitute for \( q_2 \),
\[
\pi u(q_1, x_0) - \pi u(\beta_2[q_1, x_0 + x_1 - \alpha_2]q_1, x_0/\beta_1[q_1, x_0 + x_1] - \alpha_1)
\]
\[
= (1 - \pi)v(\beta_2 x_0 + \beta_1 x_1/q_1) - (1 - \pi)v(\beta_1 x_0 + \beta_1 x_1/q_1)
\]  
(A.23)

From (19) and (A.20)–(A.22), we can then solve for \( p_6, p_1, \) and \( p_2 \) in terms of \( q_1 \) as follows:
\[
p_6 = (\beta_1/\beta_2)[(1 - \pi)x_1 - \pi q_1 x_0]/\pi[1 - (\beta_1/\beta_2)]q_1 x_0,
\]  
(A.24)
\[
p_1 = [(1 - \pi)x_1 - \pi q_1 x_0]/(1 - \pi)[1 - (\beta_1/\beta_2)](q_1, x_0 + x_1),
\]  
(A.25)
\[
p_2 = 1 - p_1 - p_6.
\]  
(A.26)

Equilibrium \( \{1, 2, 6\} \) is the solution to (A.22)–(A.26) for \( q_1, q_2, p_1, p_2, \) and \( p_6 \).

**Proposition A.4.** Equilibrium \( \{1, 2, 6\} \) is unique (within its class), if it exists, and it exists if and only if
\[
\pi u(q_1, x_0) - \pi u(\beta_2[q_1, x_0 + x_1 - \alpha_2]q_1, x_0/\beta_1[q_1, x_0 + x_1] - \alpha_1)
\]
\[
> (1 - \pi)v(\beta_2 x_0 + \beta_1 x_1/q_1) - (1 - \pi)v(\beta_1 x_0 + \beta_1 x_1/q_1)
\]  
(A.27)

and
\[
\pi u(\bar{q}_1, x_0) - \pi u(\beta_2[\bar{q}_1, x_0 + x_1 - \alpha_2]\bar{q}_1, x_0/\beta_1[\bar{q}_1, x_0 + x_1] - \alpha_1)
\]
\[
< (1 - \pi)v(\beta_2 x_0 + \beta_1 x_1/\bar{q}_1) - (1 - \pi)v(\beta_1 x_0 + \beta_1 x_1/\bar{q}_1),
\]  
(A.28)

where
\[
q_1 \equiv \{(1 - \pi)^2 \beta_1 x_1 + [(1 - \pi)^4 \beta_1^2 x_1^2 + 4\pi \beta_2(1 - \pi)^2 \beta_1^2 x_1^2]^{1/2}\}/2\pi \beta_2 x_0
\]  
(A.29)

and
\[
\bar{q}_1 \equiv (1 - \pi)x_1/\pi x_0.
\]  
(A.30)

**Proof.** As (21) holds for \( j = 1 \) and \( k = 2 \), we have \( \beta_2/q_2 > \beta_1/q_1 \), which implies that (21) holds for \( j = 6 \) and \( k = 7 \). In addition, (A.20) implies that \( q_1 > 0 \), and (21) holds for \( j = 2 \) and \( k = 3, 4, 5 \).
For a solution to eq. (A.23) to exist, denoted \( q_1^* \), it is necessary and sufficient (by inspection) that

\[
\beta_2 > \beta_1.
\]  
(A.31)

Given (A.31), the right side of (A.23) is increasing in \( q_1 \) since \( v(\cdot) \) is concave. Also, given (A.31), the left side of (A.23) is decreasing in \( q_1 \) for \( q_1 = q_1^* \), given that \( u(\cdot) \) is concave and \(-cu''(c)/u'(c) \geq 1\). Thus, given (A.31), \( q_1^* \) is unique. Given \( q_1^* \), (A.22) and (A.24)–(A.26) solve uniquely for \( q_2, p_1, p_2, \) and \( p_6 \). We need to check that \( p_1 > 0, p_2 > 0, \) and \( p_6 > 0 \). From (A.24)–(A.26), these conditions are satisfied if and only if \( q_1^* > q_1 \) and \( q_1^* < q_1 \). Therefore, from (A.23), a unique equilibrium exists if and only if (A.27) and (A.28) hold. Q.E.D.

**Equilibrium \( \{1, 6, 8\} \)**

Here, \( p_1 > 0, p_6 > 0, p_8 > 0, \) and \( p_j = 0 \) for \( j = 2, 3, 4, 5, 7 \). From (17), we get

\[
q_1 = (1 - \pi)p_6 x_1/p_1 x_0,  \tag{A.32}
\]

and, from (18) and (A.32),

\[
q_2 = (1 - \pi)(p_1 + p_8)(x_1 - x_2)/\pi(p_6 + p_8)x_0
+ (1 - \pi)^2 p_6 x_1/\pi(p_6 + p_8)x_0. \tag{A.33}
\]

Given \( p_6 > 0 \) and \( p_8 > 0 \), (21) must hold with equality for \( j = 6 \) and \( k = 8 \), that is, using (14) and (16),

\[
\beta_1 x_1/q_1 = \beta_2 (x_1 - x_2)/q_2.  \tag{A.34}
\]

Similarly, (21) holds with equality for \( j = 1 \) and \( k = 6 \), which gives, using (9) and (14), using (A.34) to substitute for \( q_2 \), and rearranging,

\[
\pi u(q_1 x_0) - \pi u(\beta_2 (x_1 - x_2) q_1 / \beta_1 x_1 - x_1)
= -(1 - \pi) v(\beta_1 x_1 x_0/[x_1 - x_2] + \beta_1 x_1/q_1)
+ (1 - \pi) v(\beta_2 x_0 + \beta_1 x_1/q_1). \tag{A.35}
\]
From (19), (A.32), (A.33), and (A.34), we can solve for $p_1$, $p_6$, and $p_8$, given $q_1$, as follows:

\[
p_6 = \frac{[(1 - \pi)(x_1 - \alpha_2) - q_1 x_0 \beta_2 \pi (x_1 - \alpha_2) / \beta_1 x_1]}{[(1 - \pi)(x_1 - \alpha_2) - (1 - \pi) \pi \beta_2 (x_1 - \alpha_2) / \beta_1 - (1 - \pi)^2 x_1]},
\]

(A.36)

\[
p_1 = (1 - \pi) x_1 p_6 / q_1 x_0,
\]

(A.37)

\[
p_8 = 1 - p_1 - p_6.
\]

(A.38)

Equilibrium $\{1, 6, 8\}$ is the solution to (A.34)-(A.38) for $q_1$, $q_2$, $p_1$, $p_6$, and $p_8$.

**Proposition A.5.** Equilibrium $\{1, 6, 8\}$ is unique (within its class), if it exists. Further, equilibrium $\{1, 6, 8\}$ exists if and only if

\[
\pi u(q_1 x_0) - \pi u(\beta_2 (x_1 - \alpha_2) q_1 x_0 / \beta_1 x_1 - \alpha_1)
\]

\[
> - (1 - \pi) v(\beta_1 x_1 x_0 / [x_1 - \alpha_2] + \beta_1 x_1 / q_1)
\]

\[
+ (1 - \pi) v(\beta_2 x_0 + \beta_1 x_1 / q_1)
\]

(A.39)

and

\[
\pi u(\tilde{q}_1 x_0) - \pi u(\beta_2 (x_1 - \alpha_2) \tilde{q}_1 x_0 / \beta_1 x_1 - \alpha_1)
\]

\[
< - (1 - \pi) v(\beta_1 x_1 x_0 / [x_1 - \alpha_2] + \beta_1 x_1 / \tilde{q}_1)
\]

\[
+ (1 - \pi) v(\beta_2 x_0 + \beta_1 x_1 / \tilde{q}_1),
\]

(A.40)

where

\[
q_1 \equiv \beta_1 (1 - \pi) x_1 / \beta_2 \pi x_0
\]

(A.41)

and

\[
\tilde{q}_1 \equiv \{(1 - \pi)^2 x_1^2 + [(1 - \pi)^4 x_1^4 + 4 \beta_2 x_1^2 (x_1 - \alpha_2)^2 (1 - \pi)^2 \pi / \beta_1]^{1/2}\}
\]

\[
\times \beta / 2 \beta_2 (x_1 - \alpha_2) x_0 \pi.
\]

(A.42)
Proof. Given (A.34), we have

\[
U_2(q_1, q_2) = U_6(q_1, q_2) - (1 - \pi)v(\beta_1 x_1 x_0)/[x_1 - \alpha_2] + \beta_1 x_1/q_1
\]

\[
+ (1 - \pi)v(\beta_1 x_0 + \beta_1 x_1/q_1)
\]

\[
< U_6(q_1, q_2),
\]

and (21) holds for \(j = 6 \) and \(k = 2\). Since (21) then holds for \(j = 1\) and \(k = 2\), we have \(\beta_2/q_2 > \beta_1/q_1\), which implies, from (14) and (15), that (21) holds for \(j = 6\) and \(k = 7\). In addition, (A.32) implies that \(q_1 > 0\) so that, from (9), inequality (21) is satisfied for \(j = 1\) and \(k = 3, 4, 5\).

Note that a necessary and sufficient condition for a solution to (A.35) is

\[
\beta_2(x_1 - \alpha_2)/\beta_1 x_1 > 1.
\]  \hspace{1cm} (A.43)

Given (A.43), the concavity of \(u(\cdot)\) and \(v(\cdot)\) implies that the left side of (A.35) is decreasing in \(q_1\) and the right side of (A.35) is increasing in \(q_1\). Therefore, if and only if (A.43) holds, a unique solution exists to (A.35), which we will denote \(q_1^{**}\). Given \(q_1^{**}\), we can solve uniquely for \(q_2\), \(p_1\), \(p_6\), and \(p_8\), from (A.34) and (A.36)–(A.38). For this solution to be an equilibrium, we must have \(p_1 > 0\), \(p_6 > 0\), and \(p_1 + p_6 < 1\) or, using (53)–(55), \(q_1^{**} > q_1\), and \(q_1^{**} < \tilde{q}_1\). Therefore, substituting in (A.35), a unique equilibrium of type \(\{1, 6, 8\}\) exists if and only if (A.39 and (A.40) hold. Q.E.D.

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