Credit in a Random Matching Model with Private Information*

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We consider a random matching model where agents have complete access to each others' histories. Exchange is motivated by risk sharing, given random unobservables incomes. There is capital accumulation and an endogenous interest rate. The key feature of this environment is that information is mobile across locations, while there are frictions associated with transporting goods. Optimal allocations in the dynamic private information environment resemble real-world credit arrangements in that there are credit balances, credit limits, and installment payments. The steady state has the property that there is a limiting distribution of expected utility entitlements with mobility and a positive fraction of agents who are credit constrained. Journal of Economic Literature Classification Numbers: D8, E1.

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1. INTRODUCTION

In this paper, we construct and study an economic environment in which intertemporal allocation, interpreted as a credit arrangement, occurs in an environment in which there are frictions associated with the movement of goods across locations, but where information flows are unrestricted. Economic agents in this environment will each consume in a sequence of separate randomly determined locations but will be able to communicate with a centralized “credit” agency at each date.

We construct a model in which each of a continuum of agents is matched randomly with a location in each period, and production and accumulation of capital take place at a centralized location. Consumption goods are shipped to each dispersed location from the central location at the beginning of each period, before any communication can take place, and these shipments of goods are subject to capacity constraints. Agents receive random endowments that are private information, and, as in the pioneering work of Townsend (1982), the desire to share risk in the presence of private information leads to a motive for intertemporal trade by tying future transfers to current transfers. Our interest here is in how the limitations on the ability to move goods affect risk sharing and how features of the solution resemble real-world credit arrangements. To be more concrete, an interpretation of the physical environment and credit arrangement we consider is that it involves a firm simultaneously engaged in production, retail sales (through a large number of distinct retail outlets), and consumer finance—for example, General Motors.1,2

Our model is most closely related to the environment studied by Green (1987), but we add spatial separation, transportation frictions, a nonnegativity constraint on consumption, capital accumulation, and some other features. The approach we take is similar to what is done in the literature on dynamic private information (e.g., Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), Atkeson and Lucas (1992, 1995), and Wang (1995)), in that we analyze the allocation problem of a social planner who seeks to construct an efficient allocation subject to the constraints implied by private information and random matching. Kocherlakota and Wallace (1998) study credit in a random matching environment with an absence-of-double-coincidence-of-wants friction, where the focus is on the residual role of money.

The economy here is very similar to that in Aiyagari (1994). The key differences are that there is random matching, and allocations are (private information) constrained efficient in the economy studied here, whereas in Aiyagari (1994) all agents were together at all dates and the market structure and borrowing constraints were exogenously imposed. As in Aiyagari (1994), there is capital accumulation in the model constructed here, which is in contrast to most dynamic private information models, with the exception of Khan and Ravikumar (1997a, 1997b).

Efficient allocations in our model can be determined by solving a set of recursive component planning problems, as in Atkeson and Lucas (1995). We use this approach to determine the characteristics of limiting distributions for this environment. We find that a limiting distribution always exists, and it exhibits mobility, i.e., individual agents are mobile within the steady-state distribution of wealth. The efficient allocation has features that resemble real-world credit systems. That is, agents have credit balances, and there are credit limits and installment payments. Furthermore, and in spite of the fact that the marginal utility of consumption is infinite at zero and that the probability of receiving a zero endowment is positive, there is a positive mass of agents who are credit-constrained in the steady state. This contrasts with results from incomplete markets models where imperfect consumption-smoothing is obtained by exogenously shutting down markets (e.g., Aiyagari, 1994). However, as in Aiyagari (1994), we find that the interest rate is less than the time preference rate and that there is capital overaccumulation relative to a public information economy.

There are two important novelties here. The first is that we study a pure credit arrangement in a random matching environment where there is no (social) role for monetary exchange. Second, we obtain a limiting distribution with mobility through an alternative to existing approaches in the literature.3 In our model, the interest rate is endogenous, and there is capital accumulation. Allowing for capital accumulation rules out equilibrium interest rates which imply degenerate limiting distributions of expected utilities where all agents converge to the upper bound on expected utilities (here, the upper bound is implied by the resource constraints which come from random matching and immobility of resources across locations). Furthermore, our economy satisfies “nonattainability of misery” as in Aiyagari and Alvarez (1995), which prevents a degenerate distribution where all

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1Another interpretation (thanks to Neil Wallace for this), which is perhaps further from what we have in mind, is the state distribution system in the former Soviet Union.

2The “firm” in our model earns zero profits, which is not the case for General Motors, but this is not critical.

3Some fixed-interest-rate private-information economies have the property that the expected discounted utility of an arbitrarily large fraction of the population eventually becomes arbitrarily low (e.g., Green, 1987), and there are related endogenous interest rate economies (e.g., Atkeson and Lucas, 1992) where the wealth distribution continues to fan out over time. Nondegenerate limiting distributions of expected utilities with mobility are obtained by Atkeson and Lucas (1995) and Phelan (1995) by imposing a lower bound on expected utilities. In Atkeson-Lucas this lower bound is arbitrary, but Phelan makes the lower bound endogenous by supposing that long-term contracts are offered by firms to workers, and that workers can leave the contractual arrangement at any time and start a new contract with another firm.
agents are at the lower bound on expected utilities (given by the nonnegativity constraint on consumption and incentive compatibility).

The remainder of the paper is organized as follows. In Section 2, we describe the model, and in Section 3 we specify the problem the social planner solves to determine efficient allocations. Section 4 reformulates the component planning problem associated with the problem in Section 3 in terms of a Bellman equation and describes a procedure for analyzing the limiting distribution of expected utilities. In Section 5, we use the Bellman equation to characterize the efficient allocation and the limiting distribution of expected utilities. Section 6 contains a discussion of the results and assumptions, and Section 7 contains a summary and conclusion. The Appendix contains proofs of various lemmas and propositions.

2. THE MODEL

The population consists of a continuum of infinite-lived agents with unit mass, each of whom has preferences given by

\[ E_0(1 - \beta) \sum_{t=0}^{\infty} \beta^t u(c_t), \]

where \( 0 < \beta < 1, \ c_t \) is consumption, and \( u(\cdot) \) is strictly increasing and strictly concave and satisfies decreasing absolute risk aversion. Assume that \( u(0) = 0 \) and \( u'(0) = \infty \). There is a continuum of dispersed locations, also with unit mass, and a distinct central location inhabited by the social planner.

At the beginning of each period, agents are matched pairwise and at random with dispersed locations, and each agent receives an endowment, \( \theta_i \), which is an i.i.d. (across agents and time) draw from a probability distribution \( F(\theta_i) \), where \( \theta_i \geq 0 \). The agent’s endowment is private information. At the beginning of period \( t \), the social planner has \( k_t \) units of capital available at the central location, where \( k_0 \) is given. Capital can be used by the planner to produce consumption goods at the beginning of the period, according to the production function \( g(k_t) \). We assume that \( g(\cdot) \) is strictly increasing, strictly concave, and continuously differentiable, with \( g(0) = 0 \), \( g'(0) = \infty \), and \( g'(\infty) < 1 \). After consumption goods are produced at the central location, goods are transported costlessly to each dispersed location. Let \( x_t^i \) denote the quantity of consumption goods transported to location \( i \in [0,1] \). We assume that there is a capacity constraint on the transportation of goods to locations, that is, \( x_t^i \leq x^* \) for all \( i \), where \( x^* > 0 \). Given the assumptions made below, this capacity constraint will be the critical restriction that limits consumption at any location in the steady state. In principle, consumption could be limited by the production technology in the steady state. However, we assume that \( \hat{k} > E(\theta_i) \)

\[ g(\hat{k}) > x^* > g(\hat{k} - \hat{k} + E(\theta_i)), \]

(1)

where \( \hat{k} \) is the solution to

\[ g'(\hat{k}) = \frac{1}{\beta}. \]

(2)

The first inequality in (1) is designed to guarantee that transfers to agents are limited by transportation capacity and not by available output. This assumption makes it possible to separate the problem of optimal capital accumulation from that of optimal transfers to agents and makes the problem tractable. The second inequality in (1) implies that complete insurance can be an outcome under full information with respect to agents’ endowments. This inequality is a sufficient condition for transportation capacity to be nonbinding in attaining complete insurance, and thus we know that any deviation from complete insurance will be due to private information. This is not to say that private information will not interact with the transportation capacity constraint in determining the allocation, only that the capacity constraint does not matter with full information. Assumption (1) is discussed in more detail in Section 6.

We also assume, to simplify some of the exposition, that when the social planner sends consumption goods to locations, it is not yet known which agents are matched with which locations for the period. This is a natural restriction to impose in our random matching framework, and in general it enhances the friction already inherent in the fact that there exists a transportation capacity constraint.

After consumption goods arrive at a given location and the agent receives her endowment, the agent receives a transfer (which could be negative) which is determined by the agent’s reported history (recorded with the social planner) and the agent’s report of her current endowment shock. Consumption goods cannot be moved across locations during the period, but at the end of the period any consumption goods not consumed are transported back to the social planner and converted, one for one, into capital.

The environment laid out here captures the following features of developed economies. First, some production occurs locally, but much production, particularly of physical goods, requires that these goods be transported to remote locations for consumption. Second, this movement of goods is costly, and the quantity of goods that can be transported is limited by the

\[ \text{Implicitly, we are assuming that capital can be consumed and that } g(\cdot) \text{ includes the undepreciated part of capital.} \]
infrastructure in place. Third, moving information is much easier than moving goods; in the model information moves costlessly, in that agents and the social planner can communicate without cost across locations. Fourth, incomes are subject to risk and difficult to verify, and this leads to the need for insurance; in practice economic agents self-insure in part by engaging in credit arrangements.

An interpretation of the allocation arrangement that will be studied in the remainder of the paper is that the social planner plays the role of a financial intermediary, and agents at dispersed locations will be able to smooth consumption by holding credit balances with this intermediary. When an agent’s endowment is low (high) she will tend to draw down (increase) this credit balance in order to consume more (less) in the present, and less (more) in the future. The financial intermediary is also engaged in production and the distribution of consumption goods to remote locations.

3. EFFICIENT ALLOCATIONS

To study allocation in this environment, we first examine the general allocation problem faced by the social planner in this section. In the next section, we show how this problem can be formulated in a tractable recursive form.

The social planner is given $\psi_0(w)$, the distribution of date 0 expected utilities across agents, and $k_0$, the initial capital stock. The goal of the planner is to deliver $\psi_0(w)$ to the continuum of agents in an efficient manner, given the technology and $k_0$.

Each period, when goods are produced using the current capital stock, these good are shipped out to each location. If equal quantities of consumption goods are not transported to locations in each period, this can only introduce the possibility of more randomness in agents’ consumptions, and it cannot help incentives. Thus, there is no loss from considering only allocations where $x_i^t = x$ for all $t$ and all $i \in [0, 1]$, and we can therefore drop $i$ superscripts from the subsequent analysis. Moreover, the social planner will ship the largest quantity of goods possible, as this can only relax constraints in the planner’s optimization problem. Thus, we have $x_t = \min[x, g(k_t)]$.

Given the shipments of goods to locations, agents receive transfers (specified by the social planner) contingent on their initial expected utilities and histories. Let $\{\tau_i(w_0, \theta^t)\}_{t=0}^{\infty}$ denote the sequence of transfers received by an agent given the sequence of locations she visits, where $\theta^t = \{\theta_0, \theta_1, \ldots, \theta_t\}$ denotes the agent’s history of endowment shock reports to date $t$ and $w_0$ is the agent’s date 0 expected utility entitlement. At the end of period $t$, goods not transferred to agents are shipped back to

the social planner and are converted one for one into capital, $k_{t+1}$. Thus, aggregate transfers and $k_t$ implicitly determine $k_{t+1}$.

**DEFINITION 1.** An allocation $(\tau, k)$ is a set of sequences $\{\tau_i(w_0, \theta^t)\}_{t=0}^{\infty}$, $\{k_t\}_{t=1}^{\infty}$, given $k_0$, which satisfy

$$w_0 = E_0(1 - \beta) \sum_{t=0}^{\infty} \beta^t u[\tau_t(w_0, \theta^t) + \theta_t]$$

for all $w_0$,

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} u[\theta_s + \tau_s(w_0, \theta^s)]$$

$$\geq u[\theta_s + \tau_s(w_0, \{\theta_0, \theta_1, \ldots, \theta_{t-1}, \theta^t\})]$$

$$+ E_t \sum_{s=t+1}^{\infty} \beta^{s-t} u[\theta_s + \tau_s(w_0, \{\theta_0, \theta_1, \ldots, \theta_{t-1}, \theta^t, \theta_{t+1}, \ldots, \theta_s\})]$$

for all $w_0$, for all $t$, for all $\theta^t$, and for all $(\theta_0, \theta_1, \ldots, \theta_{t-1}, \theta^t)$,

$$-\theta_t \leq \tau_t(w_0, \theta^t) \leq x_t$$

for all $w_0$, $t$, and $\theta^t$; and

$$k_{t+1} \geq 0$$

$$x_t = \min[x^*, g(k_t)]$$

for all $t$.

In the above definition, (3) is a promise-keeping constraint, (4) are temporary incentive compatibility constraints, (5) is the resource constraint, (6) is a nonnegativity constraint on capital, and (7) captures the capacity constraint on goods shipped to each location.

**DEFINITION 2.** An allocation $(\tau, k)$ attains $\psi_0$ with resource cost $z \in R$ if

$$-g(k_t) + k_{t+1} + \int \int \tau_i(w, \theta^t) d\mu(\theta^t) d\psi_0(w) \leq z,$$

for all $t$, and (3)–(7) are satisfied.

In the above definition, $\mu(\theta^t)$ is the distribution of the history $\theta^t$ over agents.
Definition 3. An allocation \((\tau, k)\) is efficient if it attains \(\psi_0\) with cost \(z\), and if there is no other allocation which attains \(\psi_0\) with cost \(z' < z\).

Now, we follow Atkeson and Lucas (1995) in decentralizing the problem of determining efficient allocations by considering component planning problems. Equivalently, this could be considered an approach to determining efficient allocations using "efficiency prices." First suppose that there is a planner at the central location who starts with the initial capital stock, \(k_0\), at the beginning of the first period. In each period, this planner produces given the existing capital stock, retains some output to accumulate capital for the succeeding period, retains an additional amount of output (denoted \(T_t\)) for transfers to consumers, and sells the remaining output, facing the sequence of intertemporal prices \(\{p_t\}_{t=0}^{\infty}\). Here, \(p_t/p_s\) denotes the relative price of period \(t\) consumption goods and period \(s\) consumption goods, and we can choose the numeraires arbitrarily. As in Atkeson and Lucas (1995), rather than work with \(\{p_t\}_{t=0}^{\infty}\), it proves more convenient to work with the price sequence \(\{q_t\}_{t=0}^{\infty}\), where \(p_0 = 1 - q_0\) and \(p_t = (1 - q_t) \prod_{s=0}^{t-1} q_s\) for \(t > 0\), with \(q_t \in (0, 1)\) for all \(t\). Also note that we can determine \(q_t\) in terms of \(\{p_t\}_{t=0}^{\infty}\), since \(d_0 = 1 - p_0\) and \(d_t = (1 - q_t)/(1 - \sum_{s=1}^{t-1} d_s)\). Furthermore, the gross interest rate in period \(t\) is \(p_t/p_{t+1} = (1 - q_t)/(1 - q_{t+1})q_t\). Thus, this planner can borrow and lend at market prices and maximizes the present discounted value of profits.

In addition to the planner at the central location, there is a planner associated with each initial expected utility entitlement \(w_t\). At the beginning of each period, the planner at the central location ships for \(t > 0\), to each dispersed location. Then, after agents have been randomly allocated to locations, the planner associated with \(w_t\) receives a report from each of the agents for whom they have responsibility and makes a transfer to each. Any consumption goods not transferred to agents are returned to the planner at the central location at the end of the period. The planner responsible for agents with initial expected utility entitlement \(w_t\) minimizes the cost of delivering \(w_t\) given the price sequence \(\{q_t\}_{t=0}^{\infty}\) and the sequence of shipments \(\{x_t\}_{t=0}^{\infty}\) from the planner at the central location. That is, she chooses \(\{\tau_t(w_0, \theta^t)\}_{t=0}^{\infty}\) to solve

\[
\min \left\{ (1 - q_0) \int \tau_0(w_0, \theta_0) \, dF(\theta_0) + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s \int \tau_t(w_t, \theta^t) \, d\mu(\theta^t) \right\},
\]

subject to (3)--(5).

Now, net transfers to agents from the central planner at the beginning of the period, and the return of goods not consumed to the central planner at the end of the period, are given by

\[
T_t = \int \int \tau_t(w, \theta^t) \, d\mu(\theta^t) \, d\psi_0(w),
\]

which will in general depend on \(\{k_t\}_{t=0}^{\infty}\) via the resource constraints (5) and (7). This dependence is taken into account by the central planner in choosing a path for capital accumulation, i.e., the planner at the central location chooses \(\{k_t\}_{t=0}^{\infty}\) given \(k_0\) to solve

\[
\max \left\{ (1 - q_0) [g(k_0) - T_0 - k_1] + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s [g(k_t) - T_t - k_{t+1}] \right\}.
\]

It is then straightforward to apply Theorem 1 in Atkeson and Lucas (1995, p. 70) to show that if there exists an allocation \((\tau, k)\), prices \(\{q_t\}_{t=0}^{\infty}\), an initial distribution of expected utility entitlements \(\psi_0(w)\), and aggregate resources \(z\) such that \((\tau, k)\) solves the above minimization and maximization problems given \(\{q_t\}_{t=0}^{\infty}\) (each planner optimizes given prices), (8) is satisfied with equality for all \(t\) (market clearing), and

\[
1 - q_0 + \sum_{t=0}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s < \infty,
\]

then \((\tau, k)\) attains \(\psi_0\) with resources \(z\) and is efficient.

A potential complication in using the above characterization of efficient allocations is that the total transfers \(\{T_t\}\) appearing in (10) may depend on \(\{k_t\}\). This makes it difficult to characterize an efficient \(\{k_t\}\) sequence. A way around this difficulty is provided by the following proposition.

Proposition 1. Let the sequence \(\{k^*_t\}_{t=0}^{\infty}\) satisfy

\[
1 + q_t + (1 - q_{t+1}) q_t g(k^*_t) = 0
\]

and

\[
g(k^*_{t+1}) > x^*,
\]

where \(\{q_t\}_{t=0}^{\infty}\) satisfies (11). Let \(\{\tau_t(w_0, \theta^t)\}_{t=0}^{\infty}\) solve (9) subject to (3)--(5) and suppose that (8) is satisfied with equality for all \(t\). Then \((\tau^*, k^*)\) attains \(\psi_0\) with resources \(z\) and is efficient.

Proof. See the Appendix.

This proposition is useful since it provides a way of characterizing an efficient sequence of capital stocks using (12), provided (13) holds, and an efficient sequence of transfers by solving problem (9) subject to (3)--(5).
Note that by virtue of (13), the resource constraint for problem (9) can be replaced by

$$-\theta_t \leq \tau_t(w_0, \theta^t) \leq x^*,$$

so that the \( \{k_t\} \) sequence no longer enters problem (9).

In the next section we confine attention to steady states and use dynamic programming methods to solve problem (9) and thereby construct stationary efficient allocations.

4. BELLMAN EQUATION AND STEADY STATE EFFICIENT ALLOCATIONS

Assume now that there are only two states, i.e., \( \theta_t \in \{0, y\} \), where \( \Pr[\theta_t = y] = \pi \), and \( \Pr[\theta_t = 0] = 1 - \pi \), with \( 0 < \pi < 1 \).\(^5\) We will confine attention to steady states, where \( q_t = q \in [\beta, 1] \) for all \( t \). Let \( k_q \) be such that

$$g'(k_q) = \frac{1}{q}. \quad (14)$$

By virtue of assumptions (1) and (2) we have \( g(k_q) > x^* \). Therefore, the sequence of capital stocks \( k_t = k_q \) satisfies the conditions (12) and (13) in Proposition 1. Now, in a steady state, the other component planning problems can be specified in recursive form in terms of the following Bellman equation, where the cost function \( V_q(w) \) is interpreted as the minimum expected present discounted value of transfers required in delivering an expected utility entitlement of \( w \) to a particular agent, given \( q \).

$$V_q(w) = \min \left\{ \left[ (1 - q) \left[ \pi \tau_1(w) + (1 - \pi)\tau_0(w) \right] \right. \right.$$

$$+ q \left[ \pi V_q(w_1(w)) + (1 - \pi) V_q(w_0(w)) \right] \left. \right\} \quad (15)$$

subject to

$$\pi(1 - \beta)u(y + \tau_1(w)) + \beta w_1(w)$$

$$+ (1 - \pi)(1 - \beta)u(\tau_0(w)) + \beta w_0(w) = w, \quad (16)$$

where \( w = \pi u(y) \) and \( \tilde{w} = \pi u(y + x^*) + (1 - \pi)u(x^*) \). Here, (16) is a promise-keeping constraint, (17) and (18) are incentive constraints, and (19) are the resource constraints, where \( x^* \) denotes the quantity of consumption goods shipped to each location in the steady state. The choice variables in the optimization problem on the right-hand side of the Bellman equation are \( \tau_1(w), \tau_0(w), w_1(w), \) and \( w_0(w) \). As noted earlier, the planner always has output \( g(k_q) \), which exceeds the transportation capacity constraint given by \( x^* \). Hence, it is \( x^* \) which appears in the resource constraints (19).

Now that we have a recursive representation of the component planning problems, we can proceed to an analysis of steady-state efficient allocations. We will consider the steady state where \( z = 0 \), that is, where the total net transfer to agents is zero. We can think of solving for the steady state as follows. Given a price \( q \), we can solve for the steady-state quantity of capital, \( k_q \), from (14). Then, we can solve (15) subject to (16)–(19) to obtain \( V_q(w), \tau_{1, q}(w), \tau_{0, q}(w), w_{1, q}(w), \) and \( w_{0, q}(w) \). This then implies a dynamic stochastic path for \( w \), and we can accordingly solve for \( \phi_q(w) \), the steady-state distribution of expected utility entitlements across agents. Then, we must have

$$g(k_q) - k_q = \int [\pi \tau_{1, q}(w) + (1 - \pi)\tau_{0, q}(w)] d\phi_q(w), \quad (20)$$

where the left-hand side of (20) is output per capita at the beginning of the period minus capital set aside at the end of the period, and the right-hand side of (20) is total transfers per capita. We can write (20) as

$$H_1(q) = H_2(q), \quad (21)$$

where

$$H_1(q) = g(k_q) - k_q,$$

$$H_2(q) = \int [\pi \tau_{1, q}(w) + (1 - \pi)\tau_{0, q}(w)] d\phi_q(w).$$

Then, we can ask whether (21) is satisfied and, if not, then try another \( q \), etc.

For the analysis, it will be convenient to rewrite the Bellman equation by changing variables. Define \( C(\cdot) \) by \( C(u(c)) = c \), that is, \( C(\cdot) = u^{-1}(\cdot) \).
We then have $C(0) = 0$ and $C'(0) = 0$. Then, let $u_1(w) = u(y + \tau_1(w))$ and $u_0(w) = u(\tau_0(w))$, so that $\tau_1(w) = C(u_1(w)) - y$, $\tau_0(w) = C(u_0(w))$, and $u(y + \tau_0(w)) = u(y + C(u_0(w)))$. We will also assume that if an agent claims the high endowment, $y$, then she must be able to show it. This assumption allows us to ignore the incentive constraint (18), so that we only need to worry about the incentive constraint (17).

With the above changes in notation and the added assumption, we can rewrite the Bellman equation as follows:

$$V_q(w) = \min \left\{ \frac{(1 - q)[\pi C(u_1(w)) + (1 - \pi)C(u_0(w)) - \pi y]}{\pi V_q(w_1(w)) + (1 - \pi)V_q(w_0(w))} \right\},$$

subject to

$$\pi[(1 - \beta)u_1(w) + \beta w_1(w)] + (1 - \pi)[(1 - \beta)u_0(w) + \beta w_0(w)] = w \quad (22)$$

$$(1 - \beta)u_1(w) + \beta w_1(w) \geq (1 - \beta)u(y + C(u_0(w))) + \beta w_0(w) \quad (23)$$

$$0 \leq C(u_1(w)) \leq y + x^* \quad 0 \leq C(u_0(w)) \leq x^*$$

$$w, w_1, w_0 \in [w, \bar{w}] \quad (24)$$

5. CHARACTERIZING THE STEADY-STATE EFFICIENT ALLOCATION

Using the procedure outlined in the previous section, we can now move to a derivation of the properties of the steady-state efficient allocation. The first step is to show that the cost function $V_q(\cdot)$ is well behaved, and then to obtain a characterization of the solution to the Bellman equation. This is done in the following proposition.

**Proposition 2.** (a) $V_q(\cdot)$ is strictly increasing, convex, and continuously differentiable.

(b) $u_1(w) \geq u_0(w), w_1(w) \geq w_0(w)$, and $u_1(w)$ and $w_1(w)$ are non-decreasing in $w$.

(c) $u_0(w) = 0, C(u_1(w)) < y, w_0(w) = w$, and $w_1(w) > w$.

(d) For $w \in (w, \bar{w}), w_0(w) < w$, and $\sup\{w: w_0(w) = w\} > w$.

**Proof.** (a) See Proposition 3.5 in Aiyagari and Alvarez (1995, p. 28).

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The assumption that $u(\cdot)$ satisfies decreasing absolute risk aversion guarantees that the constraint set defined by (22)-(24) is convex without a need to resort to lotteries.
by necessitating increases in $u_1$ and $w_1$ to satisfy the incentive constraint. Consequently, the planner will prefer to raise $u_0$ and keep $w_0$ at $w$.

We can now proceed to examine the implications for the steady-state distribution of expected utilities, $\psi(w)$, of alternative steady-state prices, $q \in [\beta, 1)$. We first consider the case $q = \beta$. This case is analyzed in detail in the Appendix, and we state the following proposition here.

**Proposition 3.** Let $q = \beta$. Then $\{w_t\} \to \bar{w}$ a.s.

**Proof.** See the Appendix.

Thus, when $q = \beta$, the limiting distribution of expected utilities is degenerate at $\bar{w}$ (see Fig. 1). Therefore, in the limit each agent receives a transfer of $x^*$ in each state. We will therefore have $H_2(q) = x^* > H_1(q)$ for $q = \beta$, where the inequality follows from (1) and (2).

We will now analyze the stochastic process of expected utility entitlements for $1 > q > \beta$ and give conditions under which $H_2(q) < H_1(q)$ for some $q$. This result, together with continuity of $H_2(\cdot)$, will establish the existence of a $q^* \in (\beta, 1)$ such that $H_2(q^*) = H_1(q^*)$. The corresponding allocations are efficient and support the stationary distribution with zero cost. The first step in this process is to establish the following proposition.

**Proposition 4.** Let $q > \beta$. Then, (i) there exists $\tilde{w} \in (\bar{w}, \tilde{w})$ such that $w_1(w) = w_0(w) < w$ for $w \in [\tilde{w}, \bar{w})$, and (ii) $w_0(w) < w$ for $w \in (\tilde{w}, \bar{w})$.

**Proof.** From Lemma (4) in the Appendix, and Lemmas (1) and (2), we then know that there exists some $\tilde{w} < \bar{w}$ such that $u_1(w) = u(y + x^*)$ and $u_0(w) = u(x^*)$ for $w \in [\tilde{w}, \bar{w}]$. Therefore, for $w \in [\tilde{w}, \bar{w}]$ the Bellman equation becomes

$$V_q(w) = \min \left\{ (1 - q)x^* + q[\pi V_q(w_1(w)) + (1 - \pi) V_q(w_0(w))] \right\},$$

subject to

$$(1 - \beta)\tilde{w} + \beta[\pi w_1(w) + (1 - \pi) w_0(w)] = w,$$

$$\beta w_1(w) \geq \beta w_0(w).$$

Since $V_q(\cdot)$ is convex, without loss of generality we can set $w_1(w) = w_0(w) = (w - (1 - \beta)\tilde{w})/\beta$, and

$$V_q(w) = (1 - q)x^* + q V_q \left( \frac{w - (1 - \beta)\tilde{w}}{\beta} \right),$$

for $w \in [\tilde{w}, \bar{w}]$. It follows that, for $w \in [\tilde{w}, \bar{w}]$, $w_1(w) = w_0(w) = (w - (1 - \beta)\tilde{w})/\beta < w$. Furthermore, if $w_0(w) = \tilde{w}$, then $w_0(w) < w$ for $w > w$. If $w_0(w) > \tilde{w}$, then the first-order necessary condition for $w_0(w)$ implies

$$V_q'(w_0(w)) \leq V_q'(w) \frac{\beta}{q(1 - \pi)} \leq V_q'(w) \frac{\beta}{q} < V_q'(w),$$

which implies that $w_0(w) < w$ by convexity.

Thus, the graphs of $w_1(w)$ and $w_0(w)$ versus $w$ look like those in Fig. 2. While the $w_1(w)$ curve need not intersect the $45^\circ$ line uniquely in $(w, \bar{w})$, there will be a smallest value $w^* > w$ such that $w_1(w^*) = w^*$. Since $w_0(w) < w$ for $w \in (w, \bar{w})$, it follows that $[w, w^*]$ and $[\tilde{w}, \bar{w})$ are the only ergodic sets. We will focus only on stationary distributions which put zero probability on $\{\tilde{w}\}$. It is obvious that there exists a unique stationary distribution for $\{w_1\}$, and it exhibits mobility. When $q > \beta$, there is a tendency for the expected utility of a given agent to drift down over time, since the planner is more patient than agents. However, the incentive structure in the optimal allocation will tend to push up (down) the expected utility of agents with low (high) expected utility entitlements when they receive a high (low) endowment.

Now it is straightforward, along the lines of Atkeson and Lucas (1995),\footnote{In particular, see Lemmas 10, 11, and 12, pp. 81–82, in Atkeson and Lucas (1995).} to show that $H_3(q)$ is a continuous function on $(\beta, 1)$ (we omit the proof for brevity). What remains is to show that, for some $q \in (\beta, 1)$, $H_2(q) > H_1(q)$. We consider two cases below. Let $x = g(k_1) - k_1$ where $g(k_1) = 1.$
FIG. 2. \( q > \beta \).

Case 1 obtains when \( x^* < \hat{x} \). Here, it is obvious that for \( q \) sufficiently close to unity we must have \( H_2(q) < H_1(q) \). This is because \( H_2(q) \leq x^* \) and \( H_1(q) = g(k_q) - k_q \rightarrow \hat{x} \) as \( q \rightarrow 1 \). This is summarized in the following proposition.

**Proposition 5.** If \( x^* < \hat{x} \), then there exists \( q \in (\beta, 1) \) such that \( H_2(q) < H_1(q) \).

**Proof.** Obvious from the discussion above.

Now suppose we have case 2, where \( x^* > \hat{x} \). Define \( \hat{w} \), \( \hat{c}_0 \), and \( \hat{\beta} \) as follows:

\[
\hat{w} = (1 - \beta)[\pi u(y + \hat{x}) + (1 - \pi)u(\hat{x})] + \beta w,
\]

\[
\pi u(y + \hat{c}_0) + (1 - \pi)u(\hat{c}_0) = \hat{w},
\]

\[
\hat{\beta} = \frac{\pi u(y + \hat{c}_0)}{\pi u(y + \hat{c}_0) + (1 - \pi)u(\hat{c}_0)}.
\]

Note that \( \hat{w} \in (w, \hat{w}) \) and \( \hat{c}_0 \in (0, \hat{x}) \). The former obtains because \( 0 < \hat{x} < x^* \) and

\[
w \pi u(y) - \pi u(y + \hat{x}) + (1 - \pi)u(\hat{x}) < \pi u(y + x^*) + (1 - \pi)u(x^*) = \hat{w}.
\]

The latter can be seen as follows. Let

\[
h(c_0) = \pi u(y + c_0) + (1 - \pi)u(c_0).
\]

Then,

\[
h(0) = w < \hat{w},
\]

\[
h(\hat{x}) = \pi u(y + \hat{x}) + (1 - \pi)u(\hat{x}) > \hat{w}.
\]

Furthermore, \( h(\cdot) \) is strictly increasing. Hence, there exists a unique \( \hat{c}_0 \in (0, \hat{x}) \) which satisfies \( h(\hat{c}_0) = \hat{w} \).

We can now state a proposition giving conditions such that \( H_2(q) < H_1(q) \) for some \( q \in (\beta, 1) \) when case 2 obtains.

**Proposition 6.** Suppose \( x^* > \hat{x} \). If \( \beta < \hat{\beta} \), then there exists \( q \in (\beta, 1) \) such that \( H_2(q) < H_1(q) \).

**Proof.** See the Appendix.

For the intuition behind this proposition, consider an analogy to a Bewley-type incomplete markets model in which a consumer can save at an interest rate of \( 1/q - 1 \). If \( q \) is sufficiently close to unity and \( \beta \) is sufficiently small, then the consumer faces a very low return on saving and does not care much for the future. Consequently, her optimal choice will be close to autarkic. In the present context, this means that total transfers will be quite small. That is, \( H_2(q) \) will be small when \( q \) is close to unity and \( \beta \) is sufficiently small.

We can now state the following proposition.

**Proposition 7.** If either (i) \( x^* < \hat{x} \) or (ii) \( x^* > \hat{x} \) and \( \beta < \hat{\beta} \) then there exists \( q^* \in (\beta, 1) \) such that \( H_2(q^*) = H_1(q^*) \).

**Proof.** Obvious.

The allocation and the stationary distribution associated with \( q^* \) are efficient and consistent with zero total net transfers from the planner.\(^8\)

### 6. DISCUSSION OF RESULTS AND ASSUMPTIONS

We obtain a limiting distribution with mobility of agents over expected utility entitlements for the following reasons. First, because of the nonnegativity constraint on consumption, and the fact that the lower bound on expected utilities is strictly preferred to consuming zero forever, the lower

\(^8\)The restriction on \( \beta \) required in case (ii) of Proposition 7 also appears in other dynamic models of insurance with private information. For example, the model of unemployment insurance in Atkeson and Lucas (1995) contains a similar restriction (see p. 81) to guarantee the existence of efficient stationary allocations which support stationary distributions with zero transfers. Furthermore, note that the condition of part (ii) of Proposition 7 can be consistent with the assumption made in (1). To see this, note that if \( \pi \) is close to unity, then \( \beta \) will be close to unity.
bound on expected utilities is not an absorbing state, and agents will in general drift this upward drift will continue until all agents reach the boundary of the expected utility constraints. The stochastic law of motion for \( w_t \) (see Fig. 2) has several features, including a credit limit, a credit ceiling, and a credit multiplier. If an agent starts with \( w_0 = w_{0} \), then payment cannot be made. If the agent receives positive income, then payment will be made. If the agent receives a credit ceiling, then payment will be made. If the agent builds up the credit limit, then he/she continues to consume. If the agent builds up the credit limit, then he/she eventually reaches the credit limit with \( w_t = w_{0} \).

Note that the steady-state distribution \( \pi(x) \) will have the property that a positive mass of agents is concentrated on the upper bound on expected utilities. This is a consequence of the expected utility model with incomplete markets. In that model, for any agent \( (x) \), the expected utility is a convex function of the endowment \( w_t \). Since \( d \) is a strictly increasing function of the endowment, the expected utility model holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large. Therefore, in a little time later, the expected utility model also holds that if the endowment is large, then the expected utility is large.
Credit in a Random Matching Model

Capacity constraint) equal total output, i.e., \( g(k) \). However, the associated cost is given by \( g(k) - (g(k) - k) = k > 0 \). Therefore, \( q \leq \beta \) can never be consistent with a steady state with zero cost. Thus, the specific assumptions we made regarding the capacity constraint \( x^* \) to guarantee \( q > \beta \) in a steady state do not taint our results concerning the nature of the stationary distribution.

The second inequality in (1) can also be motivated by a consideration of the problem under public information regarding endowments. It turns out that this inequality is a sufficient condition for the capacity constraint to be nonbinding and thereby permits the obvious full insurance outcome. This can be seen as follows. While many full-insurance steady-state distributions exist under public information, we will focus on one where individuals have identical expected utility entitlements, denoted \( w^* \). The solution in this case is then given by \( q = \beta, \ g^*(k) = 1/\beta, \ c^* + \hat{k} = g^*(k) + \eta y, \ w^* = u(c^*), \ \tau_0 = c^*, \ \text{and} \ \tau_1 = c^* - y \). Here, \( \hat{k} \) is the steady-state capital stock, \( c^* \) is the constant level of consumption (i.e., there is full insurance), and \( \tau_0 \) and \( \tau_1 \) are the transfers in the bad and good states, respectively. To make sure that the capacity constraint is not binding, we need to have \( \tau_0 = c^* \leq x^* \), that is, \( g(k) - \hat{k} + \eta y \leq x^* \), which is the second inequality in (1).

Finally, it is worth noting that even though we have assumed full commitment on the part of agents, the efficient allocations we have constructed are immune to deviations to autarky on the part of agents. To see this, consider the possible incentives on the part of a agent who receives a positive income and is required to make a payment if she reports it truthfully. Might she not be better off by refusing to make the payment and living in autarky from then on? The answer is no, and the reason is that she clearly has the option of reporting zero income and avoiding a payment and receiving an expected utility entitlement that is no worse than autarky. Since the incentive constraint guarantees that she is better off reporting truthfully and making the payment, it is clear that she cannot gain by refusing to make the payment and living in autarky from then on. Thus, the consumer has no incentive to “default” at any date under the above enforcement policy, although this is due to the fact that the low endowment state is zero. This is quite different from the models of Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992).

7. SUMMARY AND CONCLUSION

Efficient steady-state allocations in our random matching environment have features which resemble observed credit arrangements. That is, consumers visit a sequence of locations, receiving goods or transferring goods, with credit arrangements governed by a centralized credit agency. The key features of the steady-state allocation can be interpreted in terms of a credit mechanism with credit balances, credit limits, and installment payments.

A novelty of this paper is that we obtain a steady-state distribution of expected utility entitlements with mobility by modeling a capital accumulation economy with an endogenous interest rate. This ensures that the social planner is more patient than consumers in the steady state. We do not impose any bounds on expected utility entitlements to obtain this result, as is done in other work (Atkeson and Lucas (1995), Phelan (1995)). Indeed, ours is one of the first dynamic insurance economies with private information to include capital accumulation (note also Khan and Ravikumar (1997a, 1997b)). If the friction on resource movement implied by random matching turns out not to matter, then the efficient steady-state allocation in our model would be a natural and useful benchmark for comparing steady-state allocations with those obtained subject to an arbitrary market structure and borrowing constraints, as in Aiyagari (1994).

Related environments are used in Aiyagari and Williamson (1998) and Williamson (1998) to study the interaction between money and credit.

APPENDIX

Proof of Proposition 1. It is obvious that \((\tau^*, k^*) \) attains \( \psi_0 \) with resources \( z \). Suppose that \((\tau^*, k^*) \) is not efficient. Then there exists \((\tilde{\tau}, \tilde{k})\) such that

\[-g(\tilde{k}) + \tilde{k} + \tilde{T} \leq z = -g(k^*) + k^* + T^*, \]

where

\[\tilde{T} = \int \tilde{\tau}(w, \theta') d\mu(\theta') d\psi_0(w), \]
\[T^* = \int \tau^*(w, \theta') d\mu(\theta') d\psi_0(w). \]

Therefore,

\[(1 - q_0)(-g(k_0) + \tilde{k} + \tilde{T}) + \sum_{r=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s (-g(k_s) + k_{s+1} + T_s) < (1 - q_0)(-g(k_0) + k^* + T^*) \]
\[+ \sum_{r=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s (-g(k_s^*) + k_{s+1}^* + T_s^*).\]
However, by virtue of (12) we must have

\[
(1 - q_0)(-g(k_0) + \tilde{k}_1) + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s (-g(\tilde{k}_t) + \tilde{k}_{t+1})
\]

\[
\geq (1 - q_0)(-g(k_0) + k_1^*) + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s (-g(k_s^*) + k_{s+1}^*),
\]

because \(\{k_{t+1}^*\}_{t=0}^{\infty}\) attains the minimum of the expression

\[
(1 - q_0)(-g(k_0) + k_1^*) + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s (-g(k_s^*) + k_{s+1}^*).
\]

Note that in addition to the first order conditions (12), the sequence of capital stocks \(\{k_{t+1}^*\}_{t=0}^{\infty}\) also satisfies the transversality condition. This can be seen as follows. The resource constraints imply that

\[-E(\theta_t) \leq T_t^* \leq g(k_t^*).\]

Hence, (8) implies that

\[z \leq k_{t+1}^* \leq g(k_t^*) + E(\theta_t) + z.
\]

By virtue of the restrictions on \(g(\cdot)\) it follows that the sequence of capital stocks \(\{k_{t+1}^*\}_{t=0}^{\infty}\) is bounded. Furthermore, the assumption on the price sequence \(\{q_t\}_{t=0}^{\infty}\) implies that

\[\lim_{t \to \infty} (1 - q_t) \prod_{s=0}^{t-1} q_s = 0.
\]

Therefore,

\[\lim_{t \to \infty} (1 - q_t) \prod_{s=0}^{t-1} q_s k_{t+1}^* = 0,
\]

which is the relevant transversality condition. Furthermore,

\[
(1 - q_0) \int \tilde{\tau}_0(w_0, \theta_0) dF(\theta_0) + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s \int \tilde{\tau}_t(w_0, \theta') d\mu(\theta')
\]

\[
\geq (1 - q_0) \int \tau_0^*(w_0, \theta_0) dF(\theta_0)
\]

\[
+ \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s \int \tau_s^*(w_0, \theta') d\mu(\theta'),
\]

as \(\{\tau^*\}\) attains the minimum in (9) and \(\{\tilde{\tau}\}\) is a feasible choice for that problem. To check this we only need to verify that \(\{\tilde{\tau}\}\) satisfies the resource constraints for problem (9) for \(t \geq 1\) with the sequence of capital stocks \(\{k_{t+1}^*\}_{t=0}^{\infty}\). This is obvious because

\[-\theta_t \leq \tilde{\tau}_t(w_0, \theta_t) \leq \min[x^*, g(\tilde{k}_t)] \leq x^* = \min[x^*, g(k_t^*)]\]

for \(t \geq 1\) by virtue of (12). Now, integrating (27) with respect to \(\psi_0(w_0)\), we have

\[
(1 - q_0) \tilde{\tau}_0 + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s \tilde{\tau}_t \geq (1 - q_0) T_0^* + \sum_{t=1}^{\infty} (1 - q_t) \prod_{s=0}^{t-1} q_s T_t^*.
\]

However, adding (26) and (28), we see that the result contradicts (25). This contradiction establishes the desired result. 

**Characterization of the Efficient Steady-State Allocation When \(q = \beta\)**

Let \(\lambda\) denote the Lagrange multiplier associated with the promise-keeping constraint (22), and let \(\mu\) be the nonnegative multiplier associated with the incentive constraint (23). The first-order necessary condition for \(w_1(w)\) is

\[-q \pi V_\beta'(w_1(w)) + \lambda \pi \beta + \mu \beta = 0, \quad \text{if } w_1(w) \in (w, \bar{w}),\]

or

\[-q \pi V_\beta'(w_1(w)) + \lambda \pi \beta + \mu \beta > 0, \quad \text{if } w_1(w) = \bar{w}.
\]

The envelope condition is

\[\lambda = V_\beta'(w).
\]

Thus,

\[V_\beta'(w_1(w)) \leq V_\beta'(w) \frac{\beta}{q} + \frac{\mu \beta}{q \pi},\]

with equality if \(w_1(w) < \bar{w}\).

Analogously, the first-order necessary condition for \(w_0(w)\) can be written as

\[V_\beta'(w_0(w)) = V_\beta'(w) - \frac{\mu \beta}{q(1 - \pi)}, \quad \text{for } w_0(w) \in (w, \bar{w}),\]

or

\[V_\beta'(w_0(w)) > V_\beta'(w) \frac{\beta}{q} - \frac{\mu \beta}{q(1 - \pi)} \quad \text{if } w_0(w) = w.
\]
LEMMA 1. If \( V'(\bar{w}) = \infty \), then there exists \( \hat{w}_1 < \bar{w} \) such that \( u_1(w) = u(y + x^*) \) for \( w \in [\hat{w}_1, \bar{w}] \).

Proof. Suppose \( u_1(w) < u(y + x^*) \). Consider increasing \( u_1(w) \) and lowering \( w_1(w) \) in order to keep \( (1 - \beta)u_1(w) + \beta w_1(w) \) constant. This satisfies all of the constraints. The change in the objective function is given by

\[
\left[ (1 - q)C'(u_1(w)) - q\alpha V'_q(w_1(w)) \frac{1 - \beta}{\beta} \right] du_1(w).
\]

We know that when \( w = \bar{w} \), \( w_1(w) = \bar{\bar{w}} \). Then, since \( V'(\bar{w}) = \infty \), in a neighborhood of \( w = \bar{w} \) the above expression must be negative, which is a contradiction.

LEMMA 2. If \( V'_q(\bar{w}) = \infty \), then there exists \( \hat{w}_2 < \bar{w} \) such that \( u_0(w) = u(x^*) \) for \( w \in [\hat{w}_2, \bar{w}] \).

Proof. Suppose \( u_0(w) < u(x^*) \). Consider raising \( u_0(w) \) and lowering \( w_0(w) \) such that \( (1 - \beta)u_0(w) + \beta w_0(w) \) is held constant. This satisfies the promise-keeping constraint. It also satisfies the incentive constraint, as the change in the right-hand side is given by

\[
(1 - \beta)\left[ u'(y + C(u_0(w))) - 1 \right] du_0(w) < 0
\]

by concavity of \( u(\cdot) \). The change in the objective function is given by

\[
\left[ (1 - q)(1 - \pi)C'(u_0(w)) - q(1 - \pi)V'_q(w_0(w)) \frac{1 - \beta}{\beta} \right] du_0(w).
\]

We know that when \( w = \bar{w} \), \( w_0(w) = \bar{w} \) and \( V'_q(\bar{w}) = \infty \). Hence, in a neighborhood of \( w = \bar{w} \), the above expression must be negative, which is a contradiction.

LEMMA 3. If \( q = \beta \), then \( V'_q(\bar{w}) < \infty \).

Proof. Suppose to the contrary that \( V'_q(\bar{w}) = \infty \). Now, let \( \hat{w} = \max[\hat{w}_1, \hat{w}_2] \). Then, for \( w \in [\hat{w}, \bar{w}] \), we know that if \( V'_q(\bar{w}) = \infty \), then \( u_1(w) = u(y + x^*) \), \( u_0(w) = u(x^*) \), and

\[
w_1(w) = w_0(w) = \frac{w - (1 - \beta)\bar{w}}{\beta} < w.
\]

But the first-order necessary condition for \( w_1(w) \) implies

\[
V'_q(w_1(w)) = V'_q(w) \frac{\beta}{q} + \frac{\mu\beta}{q\pi} > V'_q(w),
\]

which implies \( w_1(w) > w \). This is a contradiction, hence \( V'_q(\bar{w}) < \infty \).

Proof of Proposition 3. Suppose \( w_1(\bar{w}) = \bar{w} \) for some \( \bar{w} \in (w, \bar{w}) \). Since \( w_1(w) \) is nondecreasing, it follows that \( w_1(w) = \bar{w} \) for \( w \in [\bar{w}, \bar{w}] \). Consider \( w \in (w, \bar{w}) \). If \( w_1(w) \in (w, \bar{w}) \), then the first-order conditions for \( w_1(w) \) imply that

\[
V'_q(w_1(w)) = V'_q(w) \frac{\beta}{q} + \frac{\mu\beta}{q\pi} = V'_q(w) + \frac{\mu}{\pi} > V'_q(w)
\]

Hence \( w_1(w) > w \). If \( w_1(w) = \bar{w} \), then again \( w_1(w) > w \). Finally, we have already seen that \( w_1(w) > w \). It follows that in a finite number of steps, \( \{w_i\} \to \bar{w} \).

Now suppose that \( w_1(w) < \bar{w} \) for all \( w \in (w, \bar{w}) \). Let \( w \in (w, \bar{w}) \). Then \( w_1(w) \in (w, \bar{w}) \), and the first-order condition for \( w_1(w) \) yields

\[
V'_q(w_1(w)) = V'_q(w) + \frac{\mu}{\pi}.
\]

Furthermore, \( w_0(w) < \bar{w} \). If \( w_0(w) \in (w, \bar{w}) \), then the first-order condition for \( w_0(w) \) yields

\[
V'_q(w_0(w)) = V'_q(w) \frac{\beta}{q} - \frac{\mu\beta}{q(1 - \pi)} = V'_q(w) - \frac{\mu}{1 - \pi}.
\]

If \( w_0(w) = w \), then we have

\[
V'_q(w_0(w)) \geq V'_q(w) - \frac{\mu}{1 - \pi}.
\]

Therefore,

\[
V'_q(w_0(w)) \geq V'_q(w) - \frac{\mu}{1 - \pi}.
\]
Combining (29) and (30), we have

$$
\pi V'_q(w_1(w)) + (1 - \pi) V'_q(w_0(w)) \geq V'_q(w).
$$

Hence \( \{V'_q(w)\} \) follows a submartingale which is bounded above, since \( V'_q(\bar{w}) < \infty \). Hence \( \{V'_q(w)\} \) converges (a.s.), and it converges (a.s.) to \( V'_q(\bar{w}) \). Therefore, \( \{w_i\} \to \bar{w} \) (a.s.).

**Lemma 4.** If \( q > \beta \), then \( V'_q(\bar{w}) = \infty \).

**Proof.** The first derivative with respect to \( w_0(w) \) from the minimization problem on the right-hand side of the Bellman equation is as follows:

$$
-q(1 - \pi)V'(w_0) + \lambda (1 - \pi) \beta - \mu \beta,
$$

which is less than zero if \( w_0(w) = \bar{w} \), greater than zero if \( w_0(w) = \bar{w} \), and equal to zero if \( w_0(w) \in (\bar{w}, \bar{w}) \). We know that \( w_0(\bar{w}) = \bar{w} \). By continuity we have \( w_0(w) > w \) for \( w \) in a neighborhood of \( \bar{w} \). Therefore,

$$
V'_q(w_0(w)) \leq \frac{\lambda \beta}{q} - \frac{\mu \beta}{q(1 - \pi)},
$$

with equality if \( w_0(w) < \bar{w} \). Furthermore, by the envelope condition,

$$
\lambda = V'_q(w).
$$

Hence,

$$
V'_q(w_0(w)) \leq V'_q(\bar{w}) \frac{\beta}{q}.
$$

If \( V'_q(\bar{w}) < \infty \), then setting \( w = w_0(w) = \bar{w} \), we have

$$
V'_q(\bar{w}) \leq V'_q(\bar{w}) \frac{\beta}{q} < V'_q(\bar{w}),
$$

a contradiction. Therefore, \( V'_q(\bar{w}) = \infty \).

**Proof of Proposition 6.** Fix \( w = \bar{w} \). Combining the promise-keeping constraint (22) with the incentive compatibility constraint (23) at equality, we have

$$
\hat{w} = (1 - \beta) \left[ \pi u(y + C(u_0(w))) + (1 - \pi) u(C(u_0(w))) \right] + \beta w_0(w)
$$

$$
\leq (1 - \beta) \left[ \pi u(y + C(u_0(w))) + (1 - \pi) u(C(u_0(w))) \right] + \beta \hat{w}.
$$

Hence,

$$
\pi u(y + C(u_0(w))) + (1 - \pi) u(C(u_0(w))) \geq \hat{w} = \pi u(y + \bar{\epsilon}_0) + (1 - \pi) u(\bar{\epsilon}_0).
$$

Therefore, \( C(u_0(w)) \geq \bar{\epsilon}_0 \). Since the utility function \( u(\cdot) \) is assumed to satisfy decreasing absolute risk aversion, it is easy to verify that as a consequence we have

$$
\hat{\beta} = \frac{\pi u'(y + \bar{\epsilon}_0)}{\pi u'(y + \bar{\epsilon}_0) + (1 - \pi) u'(\bar{\epsilon}_0)} \leq \frac{\pi u'(y + C(u_0(w)))}{\pi u'(y + C(u_0(w))) + (1 - \pi) u'(C(u_0(w)))}.
$$

Therefore,

$$
\hat{\beta} < \frac{\pi u'(y + C(u_0(w)))}{\pi u'(y + C(u_0(w))) + (1 - \pi) u'(C(u_0(w)))}.
$$

Furthermore,

$$
\hat{w} = (1 - \hat{\beta}) \left[ \pi u(y + C(u_0(w))) + (1 - \pi) u(C(u_0(w))) \right] + \beta w_0(w)
$$

$$
\geq (1 - \hat{\beta}) \left[ \pi u(y + C(u_0(w))) + (1 - \pi) u(C(u_0(w))) \right] + \beta \hat{w}.
$$

It follows from the definition of \( \hat{w} \) that \( C(u_0(w)) \leq \hat{x} \). Therefore, \( \tau_0 = C(u_0(w)) \leq \hat{x} \). Furthermore, since \( w_1(w) \geq w_0(w) \), we have \( u_1 \leq u(y + C(u_0(w))) \). Hence, \( C(u_1(w)) \leq y + C(u_0(w)) \) and \( \tau_1 = c_1 - y \leq C(u_0(w)) \leq \hat{x} \). It follows that total transfers associated with \( w = \bar{w} \) are no greater than \( \hat{x} \). We will now show that for \( q \) sufficiently close to unity we must have \( w_1(w) < \bar{w} \). This will guarantee that the ergodic set for the stochastic process of \( \{w_i\} \) will be contained in \( [\bar{w}, \bar{w}] \). Therefore, total transfers will be less than the transfers associated with \( w = \bar{w} \). These transfers as shown above are no greater than \( \hat{x} = g(k_1) - k_1 \). It follows that \( H_2(q) < \hat{x} \) for \( q \) sufficiently close to unity. Hence, we will have the desired result that \( H_2(q) < H_1(q) \) for some \( q \in [\beta, 1) \) because \( H_1(q) \to \hat{x} \) as \( q \to 1 \). Now, to show that for \( q \) sufficiently close to unity we must have \( w_1(w) < \bar{w} \), we proceed by contradiction. Suppose that there is \( \epsilon > 0 \) such that \( w_1(w) \geq \bar{w} \) for \( q \in (1 - \epsilon, 1) \). Note that our assumptions imply that \( 0 < \tau_0 < x^* \) and \( -y < \tau_1 < x^* \). The first obtains because \( 0 < \bar{c}_0 \leq C(u_0(w)) \leq \hat{x} < x^* \) and \( \tau_0 = C(u_0(w)) \). The second obtains because \( u_1(w) \geq u_0(w) = u(C(u_0(w))) \geq u(\bar{\epsilon}_0) > 0 \), which implies \( C(u_1(w)) > 0 \), in turn implying \( \tau_1 = C(u_1(w)) - y \leq C(u_0(w)) \leq \hat{x} < x^* \). Therefore, the resource constraints are not binding. Hence, the first-order conditions with respect to \( u_0 \) and \( u_1 \) together with the envelope condition can be used to derive the following equation:

$$
V'_q(\bar{w}) = \frac{(1 - \beta) \left[ \pi \delta C'(u_1) + (1 - \pi) C'(u_0) \right]}{(1 - q) \left[ \pi \delta + 1 - \pi \right]},
$$

where \( \delta = \frac{\pi u'(y + \bar{\epsilon}_0)}{\pi u'(y + \bar{\epsilon}_0) + (1 - \pi) u'(\bar{\epsilon}_0)} \).
where
\[ \delta = C'(u_0(w))u'(y + C(u_0(w))). \]

Furthermore, the first-order condition for \( u_1 \) combined with that for \( w_1(w) \) yields
\[
C'(u_1(w)) = \frac{(1 - \beta)qV'_e(w_1(w))}{(1 - q)\beta} \geq \frac{(1 - \beta)qV'_e(\hat{w})}{(1 - q)\beta} = q = \frac{\pi\delta C'(u_1(w)) + (1 - \pi)C'(u_0(w))}{\beta[\pi\delta + 1 - \pi]}.
\]

Dividing through by \( C'(u_1) \), substituting for \( \delta \), and rearranging, we have
\[
1 \geq \frac{q\pi u'(y + C(u_0(w))) + (1 - \pi)u'(C(u_1(w)))}{\beta[\pi u'(y + C(u_0(w))) + (1 - \pi)u'(C(u_0(w)))]} \geq \frac{q\pi u'(y + C(u_0(w)))}{\beta[\pi u'(y + C(u_0(w))) + (1 - \pi)u'(C(u_0(w)))]}.
\]

However, by virtue of (31), the above cannot hold for \( q \) sufficiently close to unity. This contradiction establishes that for \( q \) sufficiently close to unity, we must have \( w_1(w) < \hat{w} \).

REFERENCES


