

## SEARCH, LIMITED PARTICIPATION, AND MONETARY POLICY\*

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A model is developed that employs recent developments in the literature on search models of money to capture the distributional effects of monetary policy in a tractable way. Deterministic and stochastic versions of the model are studied. Money is not neutral, and these nonneutralities persist, whether or not the change in the money supply is anticipated or unanticipated. At the optimum, monetary policy is geared to correcting distortions in the search sector of the economy, while correcting for the persistent effects of past monetary policy actions.

### 1. INTRODUCTION

There is no consensus among macroeconomists concerning the role of monetary policy in the short run. However, an influential idea is that limited participation in financial markets is an important mechanism behind the short-run macroeconomic effects of central bank actions. That is, there is a component of economic activity that is centralized and interconnected. This economic activity involves financial transactions among banks and other financial intermediaries and the agents who hold the assets and liabilities of these financial intermediaries. When the central bank injects outside money into the economy, or withdraws it, the economic agents who are immediately affected by the monetary injection or withdrawal are the participants in these centralized transactions. Other economic agents, who are engaged in more decentralized and less interconnected transactions, will not be affected directly by the first-round effects of central bank actions. Thus, there will be a distributional effect of monetary injections and withdrawals, as first captured in heterogeneous agent models by Grossman and Weiss (1983) and Rotemberg (1984). This distributional effect will matter in interesting ways for the price level, nominal interest rates, aggregate output, foreign exchange rates, and the distribution of consumption across the population.

The Grossman–Weiss and Rotemberg models are not analytically tractable, because of the difficulty in tracking the distribution of wealth over time. However, Lucas (1990), Fuerst (1992), and Alvarez and Atkeson (1997), among others,

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bought tractability by modeling the distributional effect of monetary policy as occurring within a representative household. A drawback of this approach is that the household is able to undo some of the distributional effects of monetary policy. That is, if the household anticipates monetary policy, it completely offsets the distributional effects of monetary injections or withdrawals, and if central bank actions are not anticipated, they are undone at the end of the period when the household reunites. Thus, in this type of model, monetary policy will matter in the short run only if it is unanticipated, and there will be no persistence.

As a limited participation model requires a sector of the economy where decentralized trading occurs, it might seem natural to model this decentralized trading sector as involving search and monetary exchange. Much progress has been made in modeling monetary search, by Kiyotaki and Wright (1989), Trejos and Wright (1995), and Shi (1995), for example, and these models have proved useful for studying the frictions that give rise to a role for money, price determination with divisibility of goods, and other issues. However, versions of these models where money is divisible typically suffer from the same intractability problems as do the limited participation models of Grossman and Weiss and Rotemberg—tracking the distribution of wealth over time is difficult.

In the recent literature, there are two approaches to solving this tractability problem. An example of the first approach, similar to that of Lucas (1990), is in Shi (1997), where households consist of many agents, some of whom are engaged in search. Although money may be redistributed among agents in the household during a period, there is typically no change during the period in the total money balances held by the household. A second approach is that of Lagos and Wright (2005), who consider a search model of money where agents meet in alternating periods in a centralized fashion. When a centralized meeting takes place, agents optimally redistribute cash balances uniformly because current-period utility is linear in labor supply. Given either of these two approaches, the distribution of money balances will be degenerate across decision-making units in the economy. However, for our purposes the approach followed by Shi (1997) will not be useful, as this will have all of the drawbacks discussed above of representative-household limited-participation models.

The model developed here adapts Lagos and Wright (2005) by permitting decentralized exchange to occur contemporaneously with centralized exchange. In the Lagos and Wright model, all agents are either simultaneously engaged in decentralized exchange or in centralized exchange. In our model there will be distributional effects of monetary policy, in that agents who are currently making decentralized transactions at the “search location” are not affected directly by central bank injections or withdrawals of outside money during the current period. The central bank interacts with agents who are engaged in centralized trade at the “centralized location.” However, over time agents move between the search location and the centralized location, so that a given agent will be directly affected by central bank actions at randomly determined dates during his or her lifetime. In the model, money injections and withdrawals will be nonneutral, whether or not they are anticipated. A money injection alters the distribution of wealth in the current period, and this effect persists over time.

If we take the distributional effect of monetary injections and withdrawals seriously, then this will be important for how a central bank should respond to anticipated and unanticipated shocks to the economy. In practice, examples of the anticipated events that catch the attention of the Federal Reserve System are the large movements in outside money from banks to consumers that occur over weekends, the change in the behavior of banks close to the end of reserve-averaging periods,<sup>2</sup> and the end-of-year holiday season. The Fed also obviously cares about the unanticipated shocks that cause business cycle fluctuations. For our purposes, it is convenient to model anticipated and unanticipated shocks as fluctuations in aggregate productivity.

The model proves to be quite tractable, both in deterministic and stochastic versions. In the deterministic version of the model, productivity follows an arbitrary path over time. It is straightforward to show that money is not neutral, in that a money injection at the first date will affect rates of return and output indefinitely, though these effects dissipate in the limit. Optimal monetary policy takes a fairly simple form, in terms of optimal growth rates of money at the centralized location. However, because of the distributional effects of money injections, optimal aggregate money supply growth rates follow a more complicated rule. Provided that substitution effects dominate the income effects of changes in productivity on labor supply, the optimal current aggregate money growth rate will increase with productivity in the next period, and decrease with productivity in the previous period. If the distribution effect of a money injection is sufficiently large, then the optimal money growth rate decreases with current productivity. This is quite different from what we obtain in standard monetary models, such as cash-in-advance models, in part because our model has the feature that all the critical monetary distortions occur at the search location, where there are no direct current effects of monetary injections or withdrawals. As well, monetary policy must always be correcting, at the optimum, for the lagged effects of past policy actions. We show in the deterministic version of the model how our results can be applied to the case of seasonal productivity fluctuations. Other work that analyzes seasonality and monetary policy includes Champ et al. (1996), Chatterjee (1997), Gomis-Porqueras and Smith (2003), and Lui (2000).

In the stochastic version of the model, we are able to obtain closed form solutions in some cases, and can solve for an optimal monetary policy. In standard cash-in-advance models, optimal monetary policy is geared to smoothing the intertemporal price distortions arising from fluctuations in aggregate productivity. It is typical in these models for the optimal money growth rate to be relatively high during a period when productivity is relatively high, as this will dampen price level fluctuations and eliminate distortions. Our model works quite differently, as the goal of monetary policy is to induce the appropriate quantity of production at the search location. In a period with high productivity, sellers at the search location are induced to produce a high quantity of output if buyers have a high quantity of real balances. This will be the case if the central bank sets a low money growth rate when productivity is high. Thus, in our model optimal monetary policy

<sup>2</sup> See Lacker (2003).

acts to amplify rather than dampen price level fluctuations and fluctuations in the inflation rate.

The key contributions of the article are: (i) developing a tractable model of the persistent effects of monetary policy on output, employment, and prices propagated through the distribution of wealth, and (ii) using this model to determine optimal policy. The asset pricing models studied by Alvarez et al. (2002) and Alvarez et al. (2003), which are representative of the state of the art in the limited participation literature, have persistent effects of monetary policy on asset prices, but these are models with exogenous income where the completeness of contingent claims markets implies that monetary policy does not affect the distribution of wealth. Since aggregate wealth is exogenous and monetary policy has no effect on the distribution of wealth, monetary policy has no implications for welfare in these models. A more closely related paper is Shi (2004), which is a search-theoretic model that uses the representative household paradigm, and so has the feature that the distribution of wealth is unaffected by monetary policy. Shi obtains persistent effects of an open market operation, essentially because there is a lagged nonneutrality of the open market operation arising from a transactions role for government bonds. A paper by Berentsen et al. (2004) also examines the asymmetric effects of monetary policy on agents in a quite different environment from ours, but one which also uses the simplifying device developed by Lagos and Wright (2005). Berentsen et al. are interested in the effects of uniform money transfers among a population of agents who have money balances that are different before they receive these transfers. In addition, the model constructed here has some features in common with random-relocation models, e.g., Champ et al. (1996).

The remainder of the article is organized as follows. In Section 2, the model is constructed. Then, in Section 3 the deterministic version of the model is laid out. We show that money is not neutral, and derive an optimal monetary policy. The implications for seasonal productivity fluctuations are discussed as well. In Section 4, we work with a stochastic version of the model, deriving a closed-form solution in the case of no monetary intervention, and then determining an optimal monetary policy. Finally, Section 5 contains the conclusion.

## 2. THE MODEL

There is a continuum of agents with unit mass, and time is indexed by  $t = 0, 1, 2, \dots, \infty$ . Each agent has preferences given by

$$(1) \quad E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) + x_t - l_t]$$

where  $0 < \beta < 1$ ,  $c_t$  is consumption of *market goods*,  $x_t$  is consumption of *home-produced goods*, and  $l_t$  is labor supply. Assume that  $u(\cdot)$  is strictly increasing and strictly concave with  $u(0) = 0$  and  $u'(0) = \infty$ . There are  $n$  perishable market goods indexed by  $i = 1, 2, \dots, n$ , and a fraction  $\frac{1}{n}$  of the population produces market

good  $i$  for  $i = 1, 2, \dots, n$ . The production technology allows an agent to produce  $\theta_t$  of his or her own market good for each unit of labor input, where  $\theta_t$  is a random variable that becomes known at the beginning of period  $t$ . Given our preference specification in (1),  $\theta_t$  can be interpreted as a technology shock or as a preference shock, but we will typically give it the first interpretation. Assume that there exists some  $\hat{l}(\theta)$  such that  $u[\theta\hat{l}(\theta)] - \hat{l}(\theta) = 0$  for all realizations  $\theta$  of the random variable  $\theta_t$ . Also assume that an agent producing market good  $i$  receives utility only from consuming market good  $i + 1$ , modulo  $n$ . Thus, there is an absence-of-double-coincidence problem. Let  $\alpha \equiv \frac{1}{n}$ . An agent has the technology to convert one unit of his or her market production good into  $\frac{1}{\theta_t}$  units of the perishable home-produced good that only they can consume. This last feature is added to avoid dealing with the nonnegativity constraint on labor supply, which would bind under some circumstances that we wish to study.

There are two locations at which trade occurs in each period, a search location and a centralized location. An agent at the search location in period  $t$  will be at the centralized location in period  $t + 1$ , whereas an agent who is at the centralized location in period  $t$  will be at the search location in period  $t + 1$  with probability  $\pi$  and at the centralized location in period  $t + 1$  with probability  $1 - \pi$ , where  $0 < \pi < 1$ . We will assume that the fraction of agents at the search location in period 0 is  $\frac{\pi}{1+\pi}$ , and so this fraction will be constant for all  $t$ . Agents at the centralized location in period  $t$  do not learn whether they will go to the search location or to the centralized location until the beginning of period  $t + 1$ , after they have determined their asset holdings. Thus, agents in the centralized location have no opportunity to trade after they have learned their location for the next period.

At the centralized location, there are competitive markets for the  $n$  market goods, where a given agent will be a seller of market good  $i$  and a buyer of market good  $i + 1$ , modulo  $n$ . Since agents have identical preferences, and there is a measure of  $\alpha$  agents of each type, there will be a symmetric equilibrium where the prices of each good in terms of money in a given period are identical. Thus, let  $\phi_t$  denote the price of fiat money in terms of goods in the centralized location.

As we will show, the quasi-linear preferences in (1) imply that allowing for intertemporal trade in the centralized location is irrelevant. That is, if we allow for any type of security that is traded at the centralized location in period  $t$ , with payoffs in subsequent periods at the centralized location, then in the equilibrium we study there is no trade in these securities. We will assume that trade is anonymous at the search location, which implies that there can only be exchanges of money for goods at the search location.

This model has something in common with random-relocation models, for example, Champ et al. (1996). However, in a random-relocation model, there are banking arrangements that allow agents to insure themselves against the event that they are relocated. Our model has no role for insurance.

Now, for an agent in the centralized location, let  $M_t$  denote fiat money balances at the beginning of period  $t$ , with  $M_{t+1}$  denoting the agent's money balances at the beginning of the following period, and  $\tau_t$  the money transfer (measured in real terms) that the agent receives from the government. Then the agent's budget constraint is

$$c_t + \phi_t M_{t+1} + \theta_t x_t = \theta_t l_t + \phi_t M_t + \tau_t$$

That is, market consumption plus the real value of money balances at the end of the period plus the value of home consumption is equal to the quantity of output produced and sold during the period, plus real money balances at the beginning of the period, plus the lump-sum transfer from the government.

At the search location, there are random pairwise meetings of agents. In these meetings, agents do not have access to the histories of other agents nor to any record-keeping technology that would permit intertemporal exchange. Thus, the only exchanges at the search location are trades of market goods for money. For each agent, the probability is  $\alpha$  that the agent is a buyer of market goods in a single-coincidence meeting,  $\alpha$  that the agent is a seller of market goods in a single coincidence meeting, and  $1 - 2\alpha$  that neither agent wants the market good that the other agent could produce. In single-coincidence meetings, the seller will produce market goods in exchange for money received from the buyer, with the quantities of market goods and money exchanged determined by a take-it-or-leave-it offer by the buyer.

At the beginning of period 0, each agent has  $\bar{M}_{-1}$  units of fiat money. The government makes an identical lump-sum money transfer  $\tau_t$ , measured in units of current-period market goods, to each agent at the centralized location in period  $t$ , so that the aggregate transfer is  $\frac{\tau_t}{1+\pi}$ . That is, letting  $\bar{M}_t$  denote the aggregate quantity of money at the beginning of period  $t$ , the government budget constraint is given by

$$(2) \quad \phi_t \bar{M}_t = \phi_t \bar{M}_{t-1} + \frac{\tau_t}{1+\pi}$$

for  $t = 0, 1, 2, \dots, \infty$ .

The model contains the following key features: First, there are some transactions in the economy that are carried out in a centralized fashion, and these transactions do not require money. Second, there are some other transactions that are carried out in a decentralized fashion using money. Third, agents are sometimes uncertain concerning whether they will be engaged in centralized or decentralized transactions. Fourth, money injections by the government will initially affect directly only the agents who are engaged in centralized transactions. These key features capture some important aspects of actual economies, and we will explore the implications of these features in the next sections.

### 3. A DETERMINISTIC VERSION OF THE MODEL AND SEASONALITY

First, consider the case where the path for productivity  $\{\theta_t\}_{t=0}^{\infty}$  is known. We will be able to obtain some results for arbitrary  $\{\theta_t\}_{t=0}^{\infty}$ , and then seasonality in  $\theta_t$  is a special case. In practice, much of central bank intervention is in response to predictable events, as discussed in the Introduction. In our model, anticipated fluctuations in aggregate productivity capture this type of predictable event.

In this section, we will first set up agents' optimization problems, and then define and characterize an equilibrium. Next, we will show that money is not neutral because of a distributional effect of money injections by the central bank. Then, we will determine an optimal monetary policy, which responds to fluctuating productivity while taking account of distributional effects. Finally, we will analyze optimal policy for the special case where productivity follows an  $n$ -cycle, with  $n$  being any positive integer.

3.1. *Optimization and Equilibrium.* First, let  $W_t(m_t)$  denote the value function for an agent at the centralized location in period  $t$ , where  $m_t$  denotes the real quantity of money held at the beginning of the period. Similarly,  $V_t(m_t)$  is the value function for an agent at the search location in period  $t$ . We then have

$$W_t(m_t) = \max_{c_t, \hat{m}_t} \left\{ \begin{array}{l} u(c_t) - \frac{1}{\theta_t}(c_t + \hat{m}_t - m_t - \tau_t) \\ + \beta[\pi V_{t+1}(\rho_{t+1}\hat{m}_t) + (1 - \pi)W_{t+1}(\rho_{t+1}\hat{m}_t)] \end{array} \right\}$$

where  $\rho_t$  denotes the gross rate of return on money between period  $t - 1$  and period  $t$ , or

$$\rho_t = \frac{\phi_t}{\phi_{t-1}}$$

Therefore, we can write the above Bellman equation as

$$(3) \quad W_t(m_t) = \max_{\hat{m}_t} \left\{ \begin{array}{l} u(c_t^*) - \frac{1}{\theta_t}(c_t^* + \hat{m}_t - m_t - \tau_t) \\ + \beta[\pi V_{t+1}(\rho_{t+1}\hat{m}_t) + (1 - \pi)W_{t+1}(\rho_{t+1}\hat{m}_t)] \end{array} \right\}$$

where optimal consumption  $c_t^*$  satisfies

$$(4) \quad \theta_t u'(c_t^*) = 1$$

Now, it is important to recognize, as in Lagos and Wright (2005), that the value function in (3) is linear in  $m_t$ . Also, if the value functions are well behaved, as we will show, then there is a unique choice of  $\hat{m}_t$  (the quantity of real balances with which an agent leaves the centralized location) that solves the optimization problem on the right-hand side of (3), and this choice is independent of  $m_t$  and  $\tau_t$ . Thus, at the end of each period, all agents in the centralized location hold the same quantity of money. The fact that  $W_t(\cdot)$  is linear in  $m_t$  also implies that securities markets at the centralized location are irrelevant. If securities are traded at the centralized location, with these securities yielding payoffs in the future at the centralized location, then all agents value these future payoffs in the same way, in a manner that does not depend on the state of the world. Thus, in equilibrium there will be no trade in these securities.

In the search location, it is always optimal for each agent to produce zero home goods, and we first need to determine exchange in single coincidence meetings. Recall that there is a probability  $\alpha$  that an agent in the search location is a seller, probability  $\alpha$  that the agent is a buyer, and probability  $1 - 2\alpha$  that there is no single coincidence and the agent does not trade. Now, suppose that two agents meet at the search location and there is a single coincidence, with the seller and the buyer holding  $\bar{m}_t$  and  $\bar{m}_t$  units of real balances (measured in terms of prices at the centralized location), respectively. If the seller supplies  $q_t$  units of labor, then  $\theta_t q_t$  units of market goods are produced, and these goods are exchanged for  $d_t$  units of real money balances from the buyer. Assuming that  $q_t$  and  $d_t$  are determined by a take-it-or-leave-it offer by the buyer, then given  $d_t$  the quantity of labor supply for the seller that extracts all of the surplus from trade for the buyer is given by

$$(5) \quad q_t = \frac{\beta \rho_{t+1} d_t}{\theta_{t+1}}$$

since the seller values each unit of real balances at  $\frac{\beta \rho_{t+1}}{\theta_{t+1}}$  and suffers disutility of one unit for each unit of labor supplied. The buyer's problem is then to maximize the surplus from trading, given the constraint that he or she cannot spend more real balances than he or she has, or

$$(6) \quad \max_{d_t} \left[ u \left( \frac{\theta_t \beta \rho_{t+1} d_t}{\theta_{t+1}} \right) - \frac{\beta \rho_{t+1} d_t}{\theta_{t+1}} \right]$$

subject to

$$(7) \quad d_t \leq \bar{m}_t$$

That is, given (5) the buyer receives  $\frac{\theta_t \beta \rho_{t+1} d_t}{\theta_{t+1}}$  market consumption goods from the seller, which are consumed, and values the  $d_t$  real balances given up in exchange according to  $\frac{\beta \rho_{t+1} d_t}{\theta_{t+1}}$ . The solution to (6) subject to (7) is

$$(8) \quad q_t = \frac{\beta \rho_{t+1} \bar{m}_t}{\theta_{t+1}}$$

$$(9) \quad d_t = \bar{m}_t$$

if

$$(10) \quad \bar{m}_t \leq \frac{q_t^* \theta_{t+1}}{\beta \rho_{t+1}}$$

and

$$q_t = q_t^*$$

$$d_t = \frac{q_t^* \theta_{t+1}}{\beta \rho_{t+1}}$$

if

$$\bar{m}_t \geq \frac{q_t^* \theta_{t+1}}{\beta \rho_{t+1}}$$

Here,  $q_t^*$  solves

$$(11) \quad \theta_t u'(\theta_t q_t^*) - 1 = 0$$

If (10) holds, then the buyer will maximize his or her surplus by spending all of his or her money balances. Otherwise, the buyer has sufficient money balances to purchase the quantity  $\theta_t q_t^*$  that maximizes unconstrained surplus, and the buyer will leave the search location with some money balances. Note that  $q_t^*$  is also the quantity of labor that maximizes social surplus (assuming equal weighting of the utilities of all agents). That is, since money balances are valued in the same way by the buyer and seller, the transfer of money balances has no effect on the net surplus from trading. Thus, given that the buyer receives the net surplus from trading and maximizes this net surplus, if the buyer is unconstrained then social net surplus must be maximized. In all of the circumstances we examine (i.e., given  $\{\theta_t\}_{t=0}^{\infty}$  and the path for the money supply, and in the equilibria we study), it will be suboptimal for an agent to enter the search location with more money balances than he or she will spend if a buyer in a single coincidence match. Therefore, (10) will hold.

Since each agent in the centralized location will choose to hold the same quantity of money, all agents entering the search location will be holding the same quantity of real balances (nominal money balances multiplied by the price of money in the centralized location), which we denote  $m_t^*$ . The Bellman equation for an agent at the search location with real balances  $m_t$  is then

$$(12) \quad V_t(m_t) = \alpha \left\{ -\frac{\beta \rho_{t+1} m_t^*}{\theta_{t+1}} + \beta W_{t+1}[\rho_{t+1}(m_t + m_t^*)] \right\} \\ + \alpha \left[ u \left( \frac{\theta_t \beta \rho_{t+1} m_t}{\theta_{t+1}} \right) + \beta W_{t+1}(0) \right] + (1 - 2\alpha) \beta W_{t+1}(\rho_{t+1} m_t)$$

In Equation (12), the first of three terms is the probability that the agent is a seller multiplied by the seller's payoff, which is minus the disutility of labor supply  $q_t$ , as given by Equation (8), plus the discounted utility from carrying real balances  $m_t + m_t^*$  into the next period. The second term is the probability that the agent is a buyer multiplied by the buyer's payoff, which is the utility from consuming  $\theta_t q_t$  plus the discounted utility from carrying zero units of real balances into the following period. Finally, the third term in Equation (12) is the probability that the agent does not trade multiplied by the discounted utility from carrying  $m_t$  units

of real balances into the next period. Recall that an agent who is at the search location in the current period will be at the centralized location next period.

Let  $\hat{m}_t$  denote the money supply per agent at the centralized location (after transfers are made), and note that the aggregate money supply is given by

$$(13) \quad \bar{M}_t = \frac{1}{1+\pi}(\hat{M}_t + \pi \hat{M}_{t-1})$$

since there are  $\frac{1}{1+\pi}$  and  $\frac{\pi}{1+\pi}$  agents at the centralized location and the search location, respectively, each agent at the centralized location has  $\hat{M}_t$  units of money, and each agent at the search location has  $\hat{M}_{t-1}$  units of money. As the key money supply variable in determining the path for equilibrium quantities and prices is  $\bar{M}_t$ , we will define a *monetary policy* as a sequence of gross growth rates for  $\hat{m}_t$ ,  $\{\hat{z}_t\}_{t=0}^{\infty}$ , where

$$(14) \quad \hat{M}_t = \hat{z}_t \hat{M}_{t-1}$$

for  $t = 0, 1, 2, 3, \dots$ , with  $\hat{M}_{-1}$  given. Then note, from (13) and (14), that the gross growth rate in the aggregate money supply,  $z_t \equiv \frac{\bar{M}_t}{\bar{M}_{t-1}}$ , is given by

$$(15) \quad z_t = \hat{z}_{t-1} \left( \frac{\hat{z}_t + \pi}{\hat{z}_{t-1} + \pi} \right)$$

Note in Equation (15) that  $z_t$  depends on  $\hat{z}_t$  and on  $\hat{z}_{t-1}$ . This is because the quantity of money at the centralized location is affected by the current money injection and the previous period's money injection, whereas the quantity of money at the search location is determined only by the previous period's money injection. If  $\pi = 0$  so that all agents are in the centralized location each period, then  $z_t = \hat{z}_{t-1}$ .

**DEFINITION.** An equilibrium consists of value functions  $\{V_t(\cdot), W_t(\cdot)\}_{t=0}^{\infty}$ , real balances per agent at the centralized location (post-transfer)  $\{\hat{m}_t\}_{t=0}^{\infty}$ , and rates of return on fiat money  $\{\rho_t\}_{t=1}^{\infty}$  such that the following four conditions hold:

1.  $\hat{m}_t$  solves the optimization problem on the right-hand side of the Bellman Equation (3) given  $\{\rho_t\}_{t=1}^{\infty}$ ,  $\{V_t(\cdot), W_t(\cdot)\}_{t=0}^{\infty}$ , for  $t = 0, 1, 2, \dots$
2.  $\{V_t(\cdot), W_t(\cdot)\}_{t=0}^{\infty}$  solves the Bellman Equations (3) and (12), for  $t = 0, 1, 2, \dots$
3.  $m_t^* = \rho_t \hat{m}_{t-1}$ , on the right-hand side of (12) for  $t = 1, 2, 3, \dots$ , with  $m_0^* = \frac{\hat{m}_0}{\hat{z}_0}$ .
- 4.

$$(16) \quad \rho_t = \frac{\hat{m}_t}{\hat{m}_{t-1} \hat{z}_t}$$

for  $t = 1, 2, 3, \dots$

In the definition of equilibrium, conditions (1) and (2) state that agents behave optimally, whereas (3) and (4) summarize market clearing at the centralized location.

*3.2. Solution and Optimal Monetary Policy.* In this subsection, we first find a simple characterization of equilibrium, and then show that we can solve explicitly for an optimal monetary policy, which is a sequence of money growth factors that attain an optimal allocation. We will show in the next subsection how this can be applied to the special case of seasonal productivity.

The value functions  $W_t(\cdot)$ ,  $t = 0, 1, 2, \dots$ , are linear in the state variable, and from (12) the value functions  $V_t(\cdot)$ ,  $t = 1, 2, 3, \dots$ , are concave and differentiable. Thus a first-order condition characterizes a solution to the optimization problem on the right-hand side of (3), given (12), and this first-order condition is

$$-\frac{1}{\theta_t} + \frac{\beta^2 \pi \rho_{t+1} \rho_{t+2} \left[ \alpha \theta_{t+1} u' \left( \frac{\theta_{t+1} \beta \hat{m}_t \rho_{t+1} \rho_{t+2}}{\theta_{t+2}} \right) + 1 - \alpha \right]}{\theta_{t+2}} + \frac{\beta(1-\pi)\rho_{t+1}}{\theta_{t+1}} = 0$$

and so substituting for  $\rho_{t+1}$  and  $\rho_{t+2}$  using Equation (16) and simplifying, we obtain

$$(17) \quad -\frac{1}{\theta_t} + \frac{\beta^2 \pi \hat{m}_{t+2} \left[ \alpha \theta_{t+1} u' \left( \frac{\theta_{t+1} \beta \hat{m}_{t+2}}{\theta_{t+2} \hat{z}_{t+1} \hat{z}_{t+2}} \right) + 1 - \alpha \right]}{\hat{m}_t \theta_{t+2} \hat{z}_{t+1} \hat{z}_{t+2}} + \frac{\beta(1-\pi)\hat{m}_{t+1}}{\theta_{t+1} \hat{z}_{t+1} \hat{m}_t} = 0$$

Then (17) is a second-order difference equation that solves for  $\{\hat{m}_t\}_{t=0}^{\infty}$  given  $\{\theta_t\}_{t=0}^{\infty}$  and  $\{\hat{z}_t\}_{t=1}^{\infty}$ .

Now, we would say that money was neutral in this model if there were alternative paths for the money supply implying alternative sequences  $\{\hat{z}_t\}_{t=1}^{\infty}$  that do not alter the sequences  $\{\hat{m}_t\}_{t=0}^{\infty}$  that are solutions to the difference Equation (17). Suppose that we consider a monetary policy where there is an initial money transfer to agents at the centralized location, such that  $\bar{M}_0 = \gamma M_{-1}$ , with zero transfers in every future period, so that the aggregate stock of money is fixed for all  $t = 0, 1, 2, 3, \dots$ , and  $\gamma > 0$  governs the level of the aggregate money stock. That is, since each agent at the centralized location initially has  $\gamma M_{-1}$  units of money, and each agent at the search location initially has  $M_{-1}$  units of money, the aggregate stock of money is

$$\bar{M}_t = \left( \frac{\gamma + \pi}{1 + \pi} \right) M_{-1}$$

for all  $t$ . Though the aggregate stock of money is constant for all  $t$ , the per capita quantity of money at each location is not constant. Given the movements of agents between locations over time, the quantity of money per capita at the centralized location is

$$(18) \quad \hat{M}_t = (1 - \pi) \hat{M}_{t-1} + \pi \hat{M}_{t-2}$$

for  $t = 2, 3, 4, \dots$ , with  $\hat{M}_0 = \gamma M_{-1}$  and  $\hat{M}_1 = [\gamma(1 - \pi) + \pi] M_{-1}$ . Thus, changes in  $\gamma$  represent one-time changes in the level of the aggregate money stock, which in typical representative-agent monetary models would be neutral. However, in this model, a change in  $\gamma$  affects the fraction of money balances held per capita in each location, and this distribution effect will persist given (18).

Next, to demonstrate the nonneutrality of money we need only to construct an example where neutrality does not hold. Suppose that  $\theta_t = 1$  for all  $t$  and  $u(c) = 2c^{\frac{1}{2}}$ , and consider the case where there is no transfer in period 0, that is,  $\gamma = 1$ . Then,  $\hat{z}_t = 1$  for  $t = 1, 2, 3, \dots$ , and from (17) one equilibrium is  $\hat{m}_t = \hat{m}$  for all  $t$ , where

$$(19) \quad \hat{m} = \frac{\beta^3 \pi^2 \alpha^2}{[1 - \beta(1 - \pi) - \beta^2(1 - \alpha)\pi]^2}$$

Now, for arbitrary  $\gamma$ , for  $t = 0$  in (17) using (18) and (19), a necessary condition for  $\hat{m}_t = \hat{m}$  to be an equilibrium is

$$\begin{aligned} & -1 + [1 - \beta(1 - \pi) - \beta^2(1 - \alpha)\pi] \left[ \frac{\gamma}{\gamma[(1 - \pi)^2 + \pi] + \pi(1 - \pi)} \right]^{\frac{1}{2}} \\ & + \frac{\beta^2 \pi(1 - \alpha)\gamma}{\gamma[(1 - \pi)^2 + \pi] + \pi(1 - \pi)} + \frac{\beta(1 - \pi)\gamma}{\gamma(1 - \pi) + \pi} = 0 \end{aligned}$$

which can be shown to hold if and only if  $\gamma = 1$ . Thus, money is not neutral here.

Money is not neutral as there is a distributional effect of the money injection in period 0. That is, agents at the centralized location receive the initial money transfer whereas agents at the search location do not. As some agents who receive the transfer at the centralized location will remain there in period 1, the price level will tend to rise in the centralized location in period 1, and this implies a decrease in the value of any real balances carried into period 1 by agents who are at the search location in period 0. The price level will tend to increase in period 1 less than in proportion to the increase in per capita money balances for the agents who initially receive it, so that agents who receive the initial money transfer receive an increase in wealth. Buyers in the search location in period 0, who have suffered a decrease in wealth, will tend to consume less as a result, whereas buyers in the search location in period 1, who have received an increase in wealth, will tend to consume more. There is a persistent effect on the distribution of money balances across locations, and money growth rates at each future date are also affected. In general,  $\gamma$  (the size of the initial money injection) will affect rates of return on money and output per match at the search location in each period, though the effects will ultimately go away, as in the limit agents will hold the same quantity of money irrespective of their location. Note that, in contrast to some limited participation models, such as Fuerst (1992), the nonneutrality of money does not depend on the money injection being unanticipated. The models of Grossman and Weiss (1983) and Rotemberg (1984) have persistent wealth distribution effects of

central bank money injections, but tractability in these two setups is very limited relative to what can be obtained here.

Now, we want to find an optimal monetary policy for the general case. In equilibrium, consumption and aggregate labor supply at the centralized location are efficient, but this is not the case at the search location. The efficient quantity of labor supply  $q_t^*$  in a single-coincidence match at the search location is given by (11), and in general we will have  $q_t \leq q_t^*$  in equilibrium. If we can find a monetary policy  $\{\hat{z}_t\}_{t=1}^\infty$  that supports  $q_t = q_t^*$  for  $t = 0, 1, 2, 3, \dots$ , in a competitive equilibrium, then this policy is optimal. Suppose that such a policy exists. Then, in (17) we have

$$q_t = \frac{\beta \hat{m}_{t+1}}{\theta_{t+1} \hat{z}_t \hat{z}_{t+1}}$$

which implies that, at the optimum,

$$(20) \quad \hat{m}_t = \frac{\theta_t \hat{z}_{t-1} \hat{z}_t q_{t-1}^*}{\beta}$$

Then, substituting in (17) for  $\hat{m}_t$ ,  $\hat{m}_{t+1}$ , and  $\hat{m}_{t+2}$  using (20) and then substituting using (11), we get

$$(21) \quad -1 + \frac{\beta^2 \pi q_{t+1}^*}{\hat{z}_{t-1} \hat{z}_t q_{t-1}^*} + \frac{\beta(1-\pi)q_t^*}{\hat{z}_{t-1} q_{t-1}^*} = 0$$

Now, note that if we substitute

$$(22) \quad \hat{z}_t = \beta \frac{q_{t+1}^*}{q_t^*}$$

in (21), then this gives a solution, so that a monetary policy  $\{\hat{z}_t\}_{t=1}^\infty$  where  $\hat{z}_t$  is given by (22) for  $t = 1, 2, 3, \dots$ , is optimal.

Recall that  $\{\hat{z}_t\}_{t=1}^\infty$  is the sequence of gross money growth rates of per capita nominal money balances at the centralized location. Clearly, the rule given by (22) looks like a standard type of Friedman rule. If

$$(23) \quad -c \frac{u''(c)}{u'(c)} < 1$$

so that, from (11), optimal labor supply increases with productivity, then the current money growth rate will equal the rate of time preference if there is no change in productivity between the current period and the next, and the current gross money growth rate will be greater (less) than the subjective discount factor if productivity increases (decreases) between the current period and the next. Thus, at the optimum there is a relatively large money injection in the current period if

agents at the search location in the following period will need a relatively large quantity of real balances to purchase the optimal quantity of output.

Whereas the optimal monetary policy given by (22) has features that look like standard Friedman rules, the optimal sequence of aggregate money growth factors looks more complicated. From (15) and (22), the optimal aggregate money growth factor in period  $t$  is

$$(24) \quad z_t = \beta \left( \frac{\beta q_{t+1}^* + \pi q_t^*}{\beta q_t^* + \pi q_{t-1}^*} \right)$$

In (24), note that the current optimal aggregate money growth factor is increasing in  $q_{t+1}^*$ , decreasing in  $q_{t-1}^*$ , and may be increasing or decreasing in  $q_t^*$ , depending on parameters. The reason for the difference in the optimal aggregate money growth factor and the centralized-location money growth factor in (22) is that, in order for the government to support the sequence of optimal money growth factors at the centralized location it must compensate for the distributional effect of monetary policy. That is, at the centralized location, some agents are those who returned from the search location and who last received a transfer from the government two periods previously. The monetary rule (24) essentially compensates for the transfer this group received two periods previously and the transfer this group did not receive in the previous period.

The parameter  $\pi$  affects the size and persistence of the distribution effect. If  $\pi$  is small, then there are only a small number of agents in the search location, and the distribution effect will dissipate quickly. Also, from (22) and (24), when  $\pi = 0$  then  $z_t = \hat{z}_t$ ; that is, when the distribution effect becomes unimportant, monetary policy at the aggregate level does not need to compensate for it.

The Friedman rule is typically interpreted as a monetary policy rule that, if implemented, implies an equilibrium in which the nominal interest rate on default-free debt is zero in all states of the world. Here, suppose that, in each period  $t$ , the government issues bonds, which each sell for  $s_t$  units of money, with each bond being a promise to pay one unit of fiat money in any period in the future when the holder chooses to redeem it at the centralized location. If the government issued such a bond, then an agent who acquired this bond in period  $t$  and was at the centralized location in period  $t + 1$  would redeem it then. If the agent was in the search location in period  $t + 1$ , then he or she would redeem the bond in period  $t + 2$ . Thus, the government bond has exactly the same payoffs for an agent as does fiat money, except that it cannot be used in transactions should the agent be a buyer in a single-coincidence meeting at the search location. The reason the bond is not accepted in exchange is left unexplained, but that does not matter for this argument. If zero government bonds are supplied in each period  $t$ , then the price of these bonds is determined, in a manner similar to (17) by

$$-\frac{s_t}{\theta_t} + \frac{\beta^2 \pi \hat{m}_{t+2}}{\hat{m}_t \theta_{t+2} \hat{z}_{t+1} \hat{z}_{t+2}} + \frac{\beta(1-\pi) \hat{m}_{t+1}}{\theta_{t+1} \hat{z}_{t+1} \hat{m}_t} = 0$$

Then, given (20) and (22), at the optimum we have  $s_t = 1$  and the nominal interest rate is zero for all  $t$ . Thus, under this interpretation, an optimal monetary policy is a Friedman rule.

3.3. *Seasonality.* It is straightforward to specialize our analysis to the case where the fluctuations in productivity are seasonal. That is, suppose that productivity follows an  $n$ -cycle, with  $\{\theta_t\}_{t=0}^\infty = \{\theta^1, \theta^2, \theta^3, \dots, \theta^n, \theta^1, \theta^2, \dots\}$  for  $n \geq 2$ , an integer. In this case, we could consider monetary policies where money growth rates at the centralized location follow an  $n$ -cycle, so that  $\{\hat{z}_t\}_{t=1}^\infty = \{\hat{z}^1, \hat{z}^2, \hat{z}^3, \dots, \hat{z}^n, \hat{z}^1, \hat{z}^2, \dots\}$ . Then, there, in general, exist monetary policies of this type where all variables of consequence follow  $n$ -cycles. In particular, in such an equilibrium  $\{\hat{m}_t\}_{t=0}^\infty = \{\hat{m}^1, \hat{m}^2, \hat{m}^3, \dots, \hat{m}^n, \hat{m}^1, \hat{m}^2, \dots\}$ , where from (17), the  $\hat{m}^i$ ,  $i = 1, 2, \dots, n$  are the solution to

$$(25) \quad -\frac{1}{\theta^i} + \frac{\beta^2 \pi \hat{m}^{i**} \left[ \alpha \theta^{i*} u' \left( \frac{\theta^{i*} \beta \hat{m}^{i**}}{\theta^{i**} \hat{z}^{i*} \hat{z}^{i**}} \right) + 1 - \alpha \right]}{\hat{m}^i \theta^{i**} \hat{z}^{i*} \hat{z}^{i**}} + \frac{\beta(1-\pi)\hat{m}^{i*}}{\theta^{i*} \hat{z}^{i*} \hat{m}^i} = 0$$

for  $i = 1, 2, 3, \dots, n$ , where  $i^* = i + 1$ , modulo  $n$ , and  $i^{**} = i + 2$ , modulo  $n$ . Then an optimal monetary policy is defined as in (11), (22), and (24).

For example, consider the case  $n = 2$  (productivity follows a two-cycle). Then, letting  $q^{i*}$  denote optimal labor supply in each single-coincidence meeting at the search location when  $\theta_t = \theta^i$ , for  $i = 1, 2$ , from (22) and (24), an optimal monetary policy is given, respectively, by

$$(26) \quad \hat{z}^i = \beta \frac{q^{j*}}{q^{i*}}$$

$$(27) \quad \hat{z}^i = \beta \left( \frac{\beta q^{j*} + \pi q^{i*}}{\beta q^{i*} + \pi q^{j*}} \right)$$

for  $(i, j) = (1, 2), (2, 1)$ . Note in (26) that, if (23) holds (again, the case where optimal labor supply increases with productivity) and  $\theta^1 > \theta^2$ , then money growth at the centralized location is low in high-productivity periods and high in low-productivity periods. However, this need not be the case for aggregate money growth at the optimum. From (27), if  $\beta > \pi$ , then optimal aggregate money growth is high (low) when productivity is low (high), but if  $\beta < \pi$ , then the reverse is true. Again, the distributional effect of monetary policy becomes larger as  $\pi$  increases. If  $\pi$  is sufficiently large, then a low proportionate aggregate money transfer is required to support a high money growth rate at the centralized location when productivity is low. This is an example of how monetary policy may need to compensate in extreme ways at the optimum in order to take account of the distributional effects of monetary policy.

## 4. A STOCHASTIC VERSION OF THE MODEL

In this section, we extend our results to a stochastic environment. We show that closed-form solutions for equilibrium allocations and optimal monetary policies can be obtained for the case of i.i.d. productivity shocks. As in the deterministic version of the model, there are persistent distributional effects of monetary policy, and this has important implications for how monetary policy should respond at the optimum to random productivity shocks.

Suppose that  $(\theta_t, \hat{z}_t)$  is random and follows a first-order Markov process, and let  $W(m_t; \theta_t, \hat{z}_t)$  and  $V(m_t; \theta_t, \hat{z}_t)$  denote the value functions for agents at the centralized location and search location, respectively. For an agent at the centralized location, we adapt the Bellman Equation (3) to obtain

$$(28) \quad W(m_t; \theta_t, z_t) \quad \max_{\hat{m}_t} \left\{ \begin{array}{l} u(c_t^*) - \frac{1}{\theta_t} (c_t^* + \hat{m}_t - m_t - \tau_t) \\ + \beta E_t [\pi V(\rho_{t+1} \hat{m}_t; \theta_{t+1}, \hat{z}_{t+1}) + (1 - \pi) W(\rho_{t+1} \hat{m}_t; \theta_{t+1}, \hat{z}_{t+1})] \end{array} \right\}$$

At the search location, in single-coincidence meetings the quantities of labor  $q_t$  supplied by the seller and real balances  $d_t$  exchanged by the buyer for  $\theta_t q_t$  goods are determined by a take-it-or-leave-it offer made by the buyer to the seller. In a manner analogous to the analysis in Section 3, suppose that the stochastic process for  $(\theta_t, \hat{z}_t)$  is such that it is optimal for the buyer in a single-coincidence meeting to leave the search location with zero money balances, so that the buyer's take-it-or-leave-it offer is always constrained by available money balances. Then, if the buyer enters the period with quantity  $\bar{m}_t$  of real balances, we have

$$(29) \quad q_t = \beta \bar{m}_t E_t \left( \frac{\rho_{t+1}}{\theta_{t+1}} \right)$$

Here, note from (28) that the marginal utility of real balances for an agent in the centralized location is  $\frac{1}{\theta_t}$ , and so the marginal utility of real balances for an agent leaving the search location is  $\beta E_t \left( \frac{\rho_{t+1}}{\theta_{t+1}} \right)$  (recall that  $\rho_t$  is the gross rate of return on money in period  $t$ ).

Then, in a manner similar to (12), we can write the Bellman equation for an agent at the search location as

$$(30) \quad V(m_t; \theta_t, z_t) = \alpha \left\{ -\beta m_t^* E_t \left( \frac{\rho_{t+1}}{\theta_{t+1}} \right) + \beta E_t W[\rho_{t+1} (m_t + m_t^*); \theta_{t+1}, \hat{z}_{t+1}] \right\} \\ + \alpha \left\{ u \left[ \theta_t m_t \beta E_t \left( \frac{\rho_{t+1}}{\theta_{t+1}} \right) \right] + \beta E_t W(0; \theta_{t+1}, \hat{z}_{t+1}) \right\} \\ + (1 - 2\alpha) \beta E_t W(m_t \rho_{t+1}; \theta_{t+1}, \hat{z}_{t+1})$$

Now, as the value function  $W(m_t; \theta_t, z_t)$  is linear in  $m_t$  and since, from (30),  $V(m_t; \theta_t, z_t)$  is concave and differentiable in  $m_t$ , the first-order condition that characterizes a solution to the optimization problem on the right-hand side of (28) is

$$(31) \quad \frac{1}{\theta_t} = \beta^2 \pi E_t \left\{ \begin{aligned} & \alpha \theta_{t+1} \rho_{t+1} E_{t+1} \left( \frac{\rho_{t+2}}{\theta_{t+2}} \right) u' \left[ \beta \theta_{t+1} \hat{m}_t \rho_{t+1} E_{t+1} \left( \frac{\rho_{t+2}}{\theta_{t+2}} \right) \right] \\ & + (1 - \alpha) \rho_{t+1} E_{t+1} \left( \frac{\rho_{t+2}}{\theta_{t+2}} \right) \end{aligned} \right\} \\ + \beta(1 - \pi) E_t \left( \frac{\rho_{t+1}}{\theta_{t+1}} \right)$$

As in the deterministic model, the equilibrium condition is given by (16), so using this equation to substitute for  $\rho_{t+1}$  and  $\rho_{t+2}$  in the first-order condition (31) and rearranging gives

$$(32) \quad \frac{\hat{m}_t}{\theta_t} = \beta^2 \pi \alpha E_t \left\{ \frac{\theta_{t+1}}{\hat{z}_{t+1}} E_{t+1} \left( \frac{\hat{m}_{t+2}}{\hat{z}_{t+2} \theta_{t+2}} \right) u' \left[ \beta \frac{\theta_{t+1}}{\hat{z}_{t+1}} E_{t+1} \left( \frac{\hat{m}_{t+2}}{\hat{z}_{t+2} \theta_{t+2}} \right) \right] \right\} \\ + \beta^2 \pi (1 - \alpha) E_t \left( \frac{\hat{m}_{t+2}}{\hat{z}_{t+1} \hat{z}_{t+2} \theta_{t+2}} \right) + \beta(1 - \pi) E_t \left( \frac{\hat{m}_{t+1}}{\hat{z}_{t+1} \theta_{t+1}} \right)$$

Now, consider the case where there is no monetary intervention, so that  $\hat{z}_t = 1$  for  $t = 0, 1, 2, \dots$ , and suppose that  $\theta_t$  is an i.i.d. random variable. Look for an equilibrium where  $\hat{m}_t = f(\theta_t)$ . Then,

$$E_t \left( \frac{\hat{m}_{t+1}}{\theta_{t+1}} \right) = A$$

where  $A > 0$  is a constant. Then, substituting in Equation (32), we obtain

$$(33) \quad \frac{\hat{m}_t}{\theta_t} = \beta^2 \pi \alpha A E_t [\theta_{t+1} u'(\beta \theta_{t+1} A)] + \beta^2 \pi (1 - \alpha) A + \beta(1 - \pi) A$$

By taking expectations on the left-hand and right-hand sides of (33) and dividing through by  $A$ , we obtain

$$1 = \beta^2 \pi \alpha E_t [\theta_{t+1} u'(\beta \theta_{t+1} A)] + \beta^2 \pi (1 - \alpha) + \beta(1 - \pi)$$

which solves for  $A$ . Note that

$$(34) \quad \hat{m}_t = \theta_t A$$

which implies that

$$q_t = A$$

Therefore labor supply is constant in this noninterventionist equilibrium in each single-coincidence meeting at the search location. Of course, this is not optimal, as labor supply should vary according to (11).

Now, suppose that we look for an optimal monetary policy in the case where  $\theta_t$  is an i.i.d. random variable. First conjecture that the optimal policy has the property that  $\hat{z}_t = g(\theta_t)$  and that the equilibrium under an optimal policy is one where  $\hat{m}_t = h(\theta_t)$ . Let  $q_t^* = q^*(\theta_t)$  denote optimal labor supply, which is the solution to (11). Then, note that

$$(35) \quad E_t \left( \frac{\hat{m}_{t+1}}{\hat{z}_{t+1}\theta_{t+1}} \right) = B$$

where  $B > 0$  is a constant. Also, after substitution in (29), we get

$$(36) \quad q^*(\theta_t) = \frac{\beta B}{\hat{z}_{t+1}}$$

Then, taking expectations on the left-hand and right-hand sides of (36), we obtain

$$(37) \quad E_t \left( \frac{1}{\hat{z}_{t+1}} \right) = \frac{\bar{q}^*}{\beta B}$$

where

$$\bar{q}^* \equiv E_t q^*(\theta_{t+1})$$

Next, substituting in (32) using (11), (35), and (37), we obtain

$$(38) \quad \frac{\hat{m}_t}{\theta_t} = \beta\pi\bar{q}^* + \beta(1-\pi)B$$

which implies that

$$E_{t-1} \left( \frac{\hat{m}_t}{\hat{z}_t\theta_t} \right) = [\beta\pi\bar{q}^* + \beta(1-\pi)B] E_{t-1} \left( \frac{1}{\hat{z}_t} \right)$$

or, using (37) and rearranging,

$$B^2 - (1-\pi)\bar{q}^*B - \pi(\bar{q}^*)^2 = 0$$

Solving this quadratic equation for  $B$ , we obtain one solution, which is

$$(39) \quad B = \bar{q}^*$$

Therefore, from (36), (38), and (39), the solution is

$$(40) \quad \hat{m}_t = \beta\theta_t\bar{q}^*$$

$$(41) \quad \hat{z}_t = \frac{\beta \bar{q}^*}{q^*(\theta_t)}$$

Then from (41), if (23) holds, then  $\hat{z}_t < \beta$  when optimal labor supply is greater than average optimal labor supply, and  $\hat{z}_t > \beta$  when the reverse is the case. Thus, in what we might consider the standard case, optimal money supply growth at the centralized location is countercyclical. This is quite different from the typical case in a standard cash-in-advance model, where optimal money growth tends to be procyclical. From (15) and (41) optimal aggregate money growth is given by

$$(42) \quad z_t = \beta \left[ \frac{\beta \frac{(\bar{q}^*)^2}{q^*(\theta_t)} + \pi \bar{q}^*}{\beta \bar{q}^* + \pi q^*(\theta_{t-1})} \right]$$

Thus, from (42), as in the deterministic version of the model, aggregate money growth needs to compensate at the optimum for the distributional effect of monetary injections, and the importance of this effect increases with  $\pi$ . Note from (42) that optimal aggregate money growth is decreasing in current productivity and in productivity lagged one period, under the assumption that (23) holds. This is quite different from what we obtained in (24) for the deterministic case, where optimal money growth was increasing in current productivity under these conditions.

In standard cash-in-advance models, optimal money growth tends to be procyclical, as optimal policy is essentially aimed at correcting intertemporal price distortions. Productivity shocks tend to make the price level countercyclical in the absence of government intervention, and so the optimal money growth rate should be high when productivity is high. For example, consider a cash-in-advance model without capital with preferences and technology identical to what is in our model.<sup>3</sup> Everyone lives at a centralized location, and there is a representative firm and a representative household. Timing during a period is such that the representative household first receives a lump-sum money transfer, works for the firm, trades money for goods with the firm (subject to cash-in-advance), and then receives wage payments in the form of money. Then equilibrium labor supply,  $q_t$ , is the solution to the Euler equation:

$$q_t = \beta E_t \left[ \frac{\theta_{t+1} q_{t+1} u'(\theta_{t+1} q_{t+1})}{z_{t+1}} \right]$$

An optimal money growth rule that yields an equilibrium solution  $q_t = q^*(\theta_t)$  for all  $t$  is

$$z_{t+1} = \beta \frac{\theta_{t+1} q^*(\theta_{t+1}) u'[\theta_{t+1} q^*(\theta_{t+1})]}{\theta_t q^*(\theta_t) u'[\theta_t q^*(\theta_t)]}$$

<sup>3</sup> Details available on request.

Therefore, with i.i.d. productivity shocks and assuming (23), optimal money growth will be procyclical.

In the i.i.d. productivity example we worked out for our model above, in the absence of government intervention the price level will be countercyclical, as derived from (34). The logic from the cash-in-advance approach above would then imply that the money growth rate should be procyclical, but in fact we have shown that  $\hat{z}_t$  is countercyclical at the optimum. Why is this? There are three reasons. First, because monetary injections occur at the centralized location, agents at the search location are not directly affected by a monetary injection in the period when it occurs. The problem here is that the inefficiencies that monetary policy needs to correct exist at the search location, not at the centralized location. Second, exchange is decentralized at the search location, whereas standard cash-in-advance models involve centralized trade and competitive equilibrium. Third, monetary policy has persistent distributional effects. At the search location each period, buyers arrive with their share of the nominal quantity of money that was present at the centralized location in the previous period. To correct the inefficiency that exists at the search location, monetary policy needs to induce sellers at the search location to supply a large (small) quantity of labor when productivity is high (low). This can be accomplished if the government injects a small (large) quantity of money at the centralized location during a high (low)-productivity period, as this will revalue real balances for buyers at the search location so that they have a relatively large (small) quantity of real balances when productivity is high (low). Thus, the appropriate policy here is for the government to amplify rather than dampen price movements at the centralized location.

From (34), the gross rate of inflation at the centralized location when there is no monetary intervention is

$$\frac{\phi_{t-1}}{\phi_t} = \frac{\hat{m}_{t-1}\hat{z}_t}{\hat{m}_t} = \frac{\theta_{t-1}}{\theta_t}$$

but at the optimum we get, from (40) and (41),

$$(43) \quad \frac{\phi_{t-1}}{\phi_t} = \frac{\hat{m}_{t-1}\hat{z}_t}{\hat{m}_t} = \frac{\beta\theta_{t-1}q^*(\theta_{t-1})}{\theta_t q^*(\theta_t)}$$

Therefore, in the case where (23) holds, so that  $q^*(\theta_t)$  is increasing in  $\theta_t$ , there will be more variability in the inflation rate at the optimum than without intervention. Thus, although long-run inflation is a concern in this model, as (43) tells us that the mean inflation rate is dictated by the rate of time preference, optimal monetary policy amplifies the variability of the inflation rate about the mean.

## 5. CONCLUSION

The model in this article pushes the search paradigm in a new direction, and permits a novel and tractable analysis of how limited participation matters for

monetary policy. In the model, some economic agents are searching for trading partners at the search location, where exchange involves bilateral trading of money for goods, whereas other agents are at the centralized location, where exchange is not subject to search frictions. Over time, agents move randomly between the two locations. When the central bank intervenes, by injecting or withdrawing outside money, this occurs at the centralized location; money is not neutral and the nonneutralities persist over time, whether the change in the money supply is anticipated or unanticipated.

The monetary search model studied by Lagos and Wright (2005) in one sense accomplishes too much. One of the goals of Lagos and Wright was to arrive at a monetary framework that would be amenable to policy analysis, but the equilibrium in the Lagos–Wright model features a degenerate distribution of money balances across the population. Therefore, if we think that an important and interesting aspect of monetary policy actions is the effects these actions have on the distribution of money balances across economic agents, then this model leaves something to be desired. However, we have shown here that the distributional effects of monetary policy can be studied in a model that retains the tractability that Lagos and Wright built into their framework.

In a deterministic setting, we arrived at a simple characterization of optimal monetary policy given an arbitrary time path for aggregate productivity, and showed how our analysis for the general case could be applied to thinking about seasonality. The seasonality case is important, as a key practical monetary policy problem is how central banks should respond to predictable shocks that occur at regular frequencies. In the stochastic version of the model, we determined closed-form solutions for the case of no policy intervention, and solved for an optimal monetary policy.

Optimal policy in this environment differs from that in standard monetary models, for example, cash-in-advance models, for two reasons. First, the distortions that monetary policy can potentially correct exist at the search location. However, the central bank cannot alter the money stock at the search location during the current period, so any current effects of monetary policy at the search location must be indirect, through prices. Second, because there are long-lived effects on the distribution of money balances from monetary injections and withdrawals, at the optimum the central bank needs to compensate for the lagged effects of its policy actions in setting policy optimally. For these reasons, the model can produce unusual policy conclusions. For example, in response to stochastic productivity shocks, it can be optimal for the central bank to amplify fluctuations in the price level and the inflation rate.

Given the model's tractability, it is potentially useful to extend it so as to address other problems, for example in asset pricing and business cycles. As well, the results in this article were all worked out under the assumption of take-it-or-leave-it offers in single coincidence meetings at the search location. Lagos and Wright (2005) emphasizes the implications of the "holdup problem" that exists when the seller in a single-coincidence meeting receives some of the surplus from exchange. This case proves much more difficult to analyze in our model, but doing so is potentially interesting.

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