Notes on Liquidity Traps with Cash-in-Advance

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Abstract

1 Model

This I think is the simplest type of setup one could use to get at the ideas. It’s a little crude, but you can find more fancy things in Williamson (2012) and Williamson (2013). For example, you can include banking, collateral constraints, and private assets that serve as collateral and in exchange, and heterogeneous agents.

There is a representative agent who maximizes

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ \theta u(c^1_t) + (1 - \theta)u(c^2_t) - \gamma n_t \right], \]

where \( 0 < \beta < 1, 0 < \theta < 1, c^1_t \) is consumption of goods which can be purchased only with money, and \( c^2_t \) is consumption of goods that can be purchased with government bonds or with money. Again, this is pretty crude, but we could put banks in the model, for example, which hold government bonds, and then bank deposits backed by government bonds are used in exchange – essentially the same thing. I’m assuming constant marginal disutility of supplying labor, \( n_t \), where \( \gamma > 0 \). I don’t think I need this, but it makes life easier.

The production technology is simple: one unit of labor produces one unit of either consumption good. Suppose that production takes place at a representative firm. The equilibrium real wage is \( w_t = 1 \).

At the beginning of period \( t \), the representative agent has \( m_t \) units of money balances, and \( b_t \) one-period government bonds (both in units of the period \( t - 1 \) consumption good 1). The agent receives the payoff on government bonds in money, trades on the asset market, and receives a lump-sum transfer \( \tau_t \) from the government. The asset market constraint for the agent is

\[ c_t + q_t b'_t + q_t b'_{t+1} + m'_t \leq \frac{p_{t-1}}{p_t} (m_t + b_t) + \tau_t, \]

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where \( q_t \) denotes the price of government bonds in terms of money, \( b'_t \) is the quantity of bonds used to purchase good 2, \( b'_{t+1} \) denotes bonds that will be held over until period \( t+1 \), and \( m'_t \) is the quantity of money used to purchase good 2. In the market for good 2,
\[
c^2_t = b'_t + m'_t,
\]
(3)
Note that bonds are not discounted when accepted in exchange, since either one bond or one unit of money is a claim to one unit of money at the beginning of period \( t+1 \), from the point of view of the firm. However, bonds can trade at a discount on the asset market, as we’ll see, i.e. we can have \( q_t < 1 \).

The agent’s budget constraint is
\[
c^1_t + q_t c^2_t + m_{t+1} + b_{t+1} + q_t b'_{t+1} = \frac{p_{t-1}}{p_t} (m_t + b_t) + \tau_t + n_t
\]
(4)
In equation (4), \( m_{t+1} \) denotes any money held over until period \( t+1 \), which may have been received as wage payments from the firm, and the quantity \( b_{t+1} \) denotes bonds received as wage payments from firms that are held over to period \( t+1 \).

The government’s budget constraint is
\[
m_t - \frac{p_{t-1}}{p_t} m_{t-1} + q_t b_t - \frac{p_{t-1}}{p_t} b_{t-1} = \tau_t
\]
(5)

2 Equilibrium with binding asset-in-advance constraints and a positive nominal interest rate

First, we’ll construct an equilibrium where constraints (2) and (3) bind, and \( q_t < 1 \). If \( q_t < 1 \), note that \( m'_t = 0 \), i.e. money will not be used to purchase good 2. As well, for now assume that \( b'_{t+1} = 0 \), and we’ll check later that this is optimal. Let \( \lambda^1_t, \lambda^2_t, \) and \( \mu_t \) denote, respectively, the multipliers associated with constraints (2) through (4). The consumer chooses \( c^1_t, c^2_t, n_t, b'_t, m_{t+1}, \) and \( b_{t+1} \), so with an interior solution (we have to worry – but not much – about the fact that we have linear disutility from labor supply) we get, respectively,
\[
\theta u'(c^1_t) - \lambda^1_t - \mu_t = 0,
\]
(6)
\[
(1 - \theta) u'(c^2_t) - \lambda^2_t - \mu_t = 0,
\]
(7)
\[
-\gamma + \mu_t = 0,
\]
(8)
\[
-g_t \lambda^1_t + \lambda^2_t = 0,
\]
(9)
\[
-\mu_t + \beta E_t \left[ \frac{p_{t-1}}{p_t} (\lambda^1_{t+1} + \mu_{t+1}) \right] = 0.
\]
(10)
Reducing (6)-(10) to something we can work with, we get
\[
-\gamma + \beta E_t \left[ \frac{p_{t-1}}{p_t} \theta u'(c^1_{t+1}) \right] = 0,
\]
(11)
\[ q_t = \frac{(1 - \theta) u'(c^2_t)}{\theta u'(c^1_t)}, \quad (12) \]
and from (11) and (12) we can write, just as in my blog post:
\[ q_t = \frac{(1 - \theta) u'(c^2_t) - \gamma}{\theta u'(c^1_t)} + \beta E_t \left[ \frac{p_{t-1} u'(c^1_{t+1})}{p_t u'(c^1_t)} \right] \quad (13) \]
\[ 1 = \frac{\theta u'(c^1_t) - \gamma}{\theta u'(c^2_t)} + \beta E_t \left[ \frac{p_{t-1} u'(c^1_{t+1})}{p_t u'(c^1_t)} \right] \quad (14) \]
The prices of bonds and money, in units of money, are on the left-hand sides of (13) and (14), respectively. The payoff on each asset is one unit of money in period \( t + 1 \), and the second term on the right-hand side of each equation is the value of that payoff. But each asset also has an associated liquidity premium, which is the first term on the right-hand side of each equation. Note that the liquidity premium for bonds (money) is the inefficiency wedge for good 2 (1) divided by the marginal utility of consumption for good 1.

Note that, for \( b^t_{t+1} = 0 \) (no bonds carried from the asset market of period \( t \) into period \( t + 1 \)), we require
\[ -q_t + \beta E_t \left[ \frac{p_{t-1} u'(c^1_{t+1})}{p_t u'(c^1_t)} \right] \leq 0, \quad (15) \]
i.e. the liquidity premium on bonds must be nonnegative in this equilibrium, from (13).

To get some intuition for what is going on in this model, construct a stationary equilibrium with a constant inflation rate, \( \frac{p_{t+1}}{p_t} = \pi \), constant consumption \( (c_1, c_2) \), and constant real money balances and real bond quantities \( m \) and \( b \). But how to specify policy? We will specify monetary policy as choosing a bond price \( q \), and then supporting that price through open market operations. A key part of the approach – and the unusual part, I think – is to specify fiscal policy as fixing the quantity of the total outstanding consolidated government debt by manipulating the path for lump-sum taxes. Fiscal policy is critical here, and this is a simple way to specify it. Let \( V \) denote the real value of the outstanding consolidated government debt, and suppose there are no bonds and currency outstanding at the beginning of period 0, so in period 0 \( \tau_0 = V \), and from the government budget constraint (5),
\[ m + q b = V, \quad (16) \]
and to support this equilibrium requires a transfer \( \tau \) in each period \( t = 1, 2, \ldots \), which from (5) is
\[ \tau = V \left( 1 - \frac{1}{\pi} \right) - \frac{(1 - q) b}{\pi}, \quad (17) \]
The first term on the right-hand side is a seignorage term – the revenue from the inflation tax, which is rebated lump sum to the representative agent, and
the second term is the negative of the interest payments on government bonds. Thus, fiscal policy fixes the real value of the consolidated government debt and then future taxes respond passively so as to support that policy, in response to whatever the central bank does. Monetary policy is thus about determining the composition of the consolidated government debt through open market operations, or choosing \( q \) and then supporting that with the appropriate asset market intervention. Fiscal policy is suboptimal here – the fiscal authority is doing something stupid, and the central bank optimizes treating the fiscal policy rule as given.

Then, in a stationary equilibrium, in which money balances are used to purchase good 1 and bonds to purchase good 2, from (11), (12), and (16), and (2) and (3) with equality, we get

\[-\gamma + \frac{\beta}{\pi} u'(c_1) = 0,\]  
\[q = \frac{(1 - \theta) u'(c_2)}{\theta u'(c_1)},\]  
\[c_1 + qc_2 = V\]

So, given \( q \) chosen by the central bank, (19) and (20) determine \( c_1 \) and \( c_2 \). Then, (18) determines the inflation rate, i.e.

\[\pi = \frac{\beta}{\gamma} u'(c_1)\]

It’s clear that suboptimal fiscal policy is constraining consumption, from (20). Indeed, to support this equilibrium we need \( V \) to be sufficiently small. If \( V \) is large enough, then there is no solution to (19) and (20) with \( 0 < q < 1 \). But that’s the whole point, as the key to what is going on in the liquidity trap has to do with the scarcity of safe assets in the aggregate.

The real rate of return on government debt in equilibrium, from (19) and (21), is

\[r = \frac{1}{q} - 1 = \frac{\gamma}{\beta(1 - \theta) u'(c_2)} - 1.\]

Thus, the real interest rate is lower than the rate of time preference, and the gap between the rate of time preference and \( r \) is increasing in the inefficiency wedge for good 2. Thus the liquidity premium on government debt is reflected in a low real rate of interest.

Further, for (15) to hold, we require

\[(1 - \theta) u'(c_2) - \gamma \geq 0,\]

i.e. the inefficiency wedge for good 2 needs to be nonnegative.

We can also consider the effects of monetary policy here, in the equilibrium with \( q < 1 \), though I’m more interested in the liquidity trap. Note that if the central bank increases \( q \), then from (19) and (20), \( c_2 \) must go down and, if the
coefficient of relative risk aversion is small enough (less than one I think) then \( c_1 \) rises and inflation falls. For example, consider the case \( u(c) = 2c^\frac{\gamma}{1+\gamma}, \) and \( \theta = \frac{1}{\gamma} \). Then, the solution is

\[
\begin{align*}
c_1 &= \frac{qV}{1+q}, \\
c_2 &= \frac{V}{q+q^2}, \\
\pi &= \frac{\beta}{2\gamma} \left( \frac{1+q}{qV} \right)^{\frac{1}{\gamma}}, \\
r &= \frac{2\gamma}{\beta u'(c_2)} \left( \frac{V}{q+q^2} \right)^{\frac{1}{\gamma}} - 1
\end{align*}
\]

Note in particular that, in the example, the inflation rate is decreasing in \( V \), and the real interest rate is increasing in \( V \), given the nominal interest rate \( \frac{1}{\gamma} - 1 \) set by the central bank. Thus, given central bank policy, if the supply of safe assets increases, this increases the real interest rate and reduces the inflation rate.

## 3 Liquidity Trap

Now, consider equilibria where \( q_t = 1 \), but where the asset-in-advance constraints bind. In this equilibrium, we want \( m'_t > 0 \), so that one type of transaction uses only money, and the other uses money and bonds. Then, \( \lambda_1^T = \lambda_2^T \). Now, if we focus on stationary equilibria, we get

\[
\begin{align*}
\pi &= \frac{\beta}{\gamma} u'(c_1), \\
u'(c_1) &= (1-\theta)u'(c_2), \\
c_1 + c_2 &= V,
\end{align*}
\]

and the central bank must set

\[
m \geq c_1
\]

i.e. money must be sufficiently plentiful so that we get the liquidity trap equilibrium, and as long as (24) holds open market purchases have no effect at the margin. The real rate of return on government debt is

\[
\frac{1}{\pi} - 1 = \frac{\gamma}{\beta u'(c_1)} - 1,
\]

so \( \frac{1}{\pi} - 1 < \frac{1}{\gamma} - 1 \), i.e. the real rate is less than what New Keynesians like to call the “Wicksellian natural rate of interest,” because of the inefficiency wedge, which is the same for both goods, i.e. \( \theta u'(c_1) - \gamma = (1-\theta)u'(c_2) - \gamma \). Why the inefficiency? Because \( V \), the supply of consolidated government debt that is useful in transactions, is too low. Further, the smaller is \( V \) the greater is
the inefficiency, the lower is consumption of both goods, and the higher is the inflation rate, from (21). Using the square root example above, in a liquidity trap,

\[ c_1 = c_2 = \frac{V}{2}, \]

\[ \pi = \frac{2^{-\frac{1}{2}} \beta V^{-\frac{1}{2}}}{\gamma} \]

\[ r = \frac{2^{\frac{1}{2}} \gamma V^{\frac{1}{2}}}{\beta u'(c_2)} - 1 \]

Note also that total output is

\[ Y = V, \]

so a more plentiful stock of safe assets reduces inflation, increases the real interest rate, and increases output in a liquidity trap. All these results apply generally.

4 Dynamics

Suppose for simplicity that all the aggregate uncertainty in this economy is coming from fiscal policy, with \( V_t \) fluctuating in some random fashion. As above, the fiscal policy rule is described by

\[ V_t = m_t + q_t b_t, \quad (25) \]

and, from the government budget constraint (5), this policy is supported by a lump-sum tax policy

\[ \tau_t = \pi_t - \frac{V_{t-1}}{\pi_t} - \left( \frac{q_{t-1} - 1}{\pi_t} \right) b_{t-1}, \]

where \( \pi_t = \frac{p_t}{p_{t-1}} \) is the gross inflation rate. From (11) and (12),

\[ -\gamma + \beta E_t \left[ \frac{\theta u'(c_{t+1})}{\pi_{t+1}} \right] = 0, \quad (26) \]

\[ q_t = \frac{(1 - \theta) u'(c_t^2)}{\theta u'(c_t^2)}, \quad (27) \]

Then, given binding asset-in-advance constraints, from (25),

\[ V_t = c_t^1 + q_t c_t^2, \quad (28) \]

Solving this is quite simple. Monetary policy sets \( q_t \), and (27) and (28) then determine \( c_t^1 \) and \( c_t^2 \). Then, we use equation (26) to determine inflation. Suppose, for example, that we are in a liquidity trap in all states of the world, and that \( V_t \in \{ V_1, V_2 \} \), where \( V_1 > V_2 \). Further, suppose that \( V_t \) is a two-state Markov
process, with \( \rho \) denoting the probability of staying in the same state, and \( 1 - \rho \) the probability of transition to the other state. Then, let \( c_i^1 \) denote consumption of good 1 in state \( i \). From (27) and (28), we know that \( c_i^1 > c_i^2 \). Then, (26) implies that

\[
\gamma = \beta \rho \frac{\theta u'(c_i^1)}{\pi_1} + \beta (1 - \rho) \frac{\theta u'(c_i^2)}{\pi_2},
\]

and the solution to those two equations is:

\[
\pi_i = \frac{\beta \theta u'(c_i^1)}{\gamma}, \quad \text{for } i = 1, 2,
\]

so inflation is high in the low-\( V \) state, and low in the high-\( V \) state. Further, note that the solution for the two-state Markov case is in fact a general solution to (26), so we can write

\[
\pi_i = \frac{\beta \theta u'(c_i^1)}{\gamma}, \quad \text{(29)}
\]

So, (26), (27), solve for \( c_i^1 \) and \( c_i^2 \), and then (29) solves for the inflation rate. An interesting feature of the equilibrium (and this is at the zero lower bound or away from it) is that, from (29), inflation is driven by the inefficiency wedge associated with good 1. If the nominal interest rate is fixed (i.e., \( q_t \) constant), inflation and \( V \) are negatively correlated.

Further, suppose we are on a deterministic path where \( V \) is increasing over time and \( q_t = 1 \). Then, from (26) and (27), consumption will be increasing over time, output and hours worked will be increasing, and from (29) inflation will be falling because the inefficiency wedge is declining over time.

### 5 Taylor Rule

Now, suppose we put into our model a Taylor-rule central banker (TRCB). The TRCB has a rudimentary understanding of Old and New Keynesian economics, and his or her Taylor rule takes the form

\[
\frac{1}{q_t} = \max[\pi_t^\alpha (\pi^*)^{1-\alpha} w g_t^{-\delta}, 1]
\]

Here, \( \frac{1}{q_t} \) is the gross nominal interest rate, \( \pi^* \) is the TRCB’s target gross inflation rate, \( w \) is what the TRCB thinks is the gross “Wicksellian natural rate of interest,” and \( g_t \) is TRCB’s perceived “output gap,” i.e. the ratio of the efficient level of output to actual output. We have \( \alpha > 1 \) and \( \delta > 0 \). [The Taylor rule is written here in multiplicative form, but you can get the usual thing if you take logs in (30) and approximate using \( \log(1 + x) \approx x \).] Thus, from (29), the TRCB will choose to set \( q_t = 1 \) (zero nominal interest rate), so long as

\[
\left[ \frac{\beta \theta u'(c_i^1)}{\gamma} \right]^\alpha (\pi^*)^{1-\alpha} w g_t^{-\delta} \leq 1 \quad \text{(31)}
\]
The TRCB is convinced that the natural rate of interest \( w \) is low – that’s his or her justification for choosing to be at the zero lower bound, along with the perceived output gap, i.e. \( g_t > 1 \). Suppose then that we are on a deterministic path on which \( V_t \) is increasing, which implies that consumption is rising, inflation is falling and, given \( \pi^* \), \( w \), and \( g_t = g \), a constant, the left-hand side of (31) is falling, so the TRCB sees no reason for “liftoff,” i.e. departure from the zero lower bound. Further the TRCB has bought into the New Keynesian Phillips curve, and thinks that falling inflation must mean that the output gap must be increasing – \( g_t \) is rising – which makes the left-hand side of (31) decrease further. As well, in a New Keynesian model, the output gap is also reflected in a departure of the natural interest rate \( w \) from the actual rate. If inflation is falling, the TRCB may also think that \( w \) is falling, which also decreases the left-hand side of (31). But what if the TRCB announces a higher inflation target? Since \( \alpha > 1 \), that also reduces the left-hand side of (31).

So, if the world works as I’ve modeled it, but monetary policy is set by TRCB, he or she gets trapped at the zero lower bound. Note that the zero lower bound is not unstable – it’s steady as a rock. Various shocks can make the left-hand side of (31) fluctuate, but you don’t get liftoff as long as the inequality holds.

6 Extensions

As I mentioned, you can see Williamson (2012, 2013) for additional bells and whistles – banks that hold reserves, private assets, collateral, etc. The models in those papers are built on a Lagos-Wright structure. That’s particularly nice for getting at credit arrangements and banking. For example, I tried doing the asset constraint as a collateral constraint in the above cash-in-advance model, and that was a very hard problem, because you have to worry about what happens off-equilibrium. I stripped down the model above, leaving out private assets, just as in Williamson (2013). I don’t think that changes any of the results. Williamson (2012) shows some of the reasons why the financial crisis could reduce the stock of private assets that serve as liquidity.

7 References


Williamson, S. 2013. “Scarce Collateral, the Term Premium, and Quantitative Easing,” working paper.