Appendix to “Monetary Policy and Distribution”

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1 Introduction
This is a technical appendix to accompany “Monetary Policy and Distribution,” currently available at http://www.artsci.wustl.edu/~swilliam/papers/money12.pdf, and under review at the Journal of Monetary Economics.

2 Equilibrium Solution With No Aggregate Uncertainty
From household optimization, we obtain the following first-order conditions,

\[ v'(n^1_t) = \beta \left\{ \frac{p^1_t(1 - (1 - \alpha)\pi )u'(c^{11}_{t+1})}{p^{1+1}_{t+1}} + \frac{p^2_t(1 - (1 - \alpha)\pi )u'(c^{12}_{t+1})}{p^{2+1}_{t+1}} \right\}, \]

(1)

\[ v'(n^2_t) = \beta \left\{ \frac{p^2_t\alpha u'(c^{21}_{t+1})}{p^{1+1}_{t+1}} + \frac{p^2_t(1 - \alpha\pi )u'(c^{22}_{t+1})}{p^{2+1}_{t+1}} \right\}, \]

(2)

and the price of the nominal bond is determined, again given household optimization, by

\[ q_t \left\{ \frac{[1 - (1 - \alpha)\pi ]u'(c^{11}_t)}{p^1_t} + \frac{(1 - \alpha)\pi u'(c^{12}_t)}{p^2_t} \right\} = \beta \left\{ \frac{[1 - (1 - \alpha)\pi ]u'(c^{11}_{t+1})}{p^{1+1}_{t+1}} + \frac{(1 - \alpha)\pi u'(c^{12}_{t+1})}{p^{2+1}_{t+1}} \right\}. \]

(3)

Market clearing gives

\[ p^i_t = \frac{q^i_t}{n^i_t}, \text{ for } i = 1, 2, \]

(4)
and consumption quantities are then determined by
\[ c_{ij}^t = n_i^t \frac{M_i^t}{\psi_i^t}, \quad \text{for } i, j = 1, 2. \]  
(5)

Then, substituting for \( p_i^t \) and \( c_{ij}^t \) for \( i, j = 1, 2 \) in equations (1)-(3) using (4) and (5) gives
\[ v_0(n_1^t) n_2^t \psi_1^t = \beta \left\{ \begin{array}{l} [1 - (1 - \alpha)\pi] u' \left( \frac{n_1^t M_1^t}{\psi_1^t} \right) n_1^t \\ + (1 - \alpha)\pi u' \left( \frac{n_2^t M_2^t}{\psi_1^t} \right) n_2^t \end{array} \right\}, \]  
(6)

\[ \frac{v'(n_2^t)}{\psi_2^t} = \beta \left\{ \begin{array}{l} \alpha\pi u' \left( \frac{n_1^t M_1^t}{\psi_1^t} \right) n_2^t \\ + (1 - \alpha)\pi u' \left( \frac{n_2^t M_2^t}{\psi_2^t} \right) n_2^t \end{array} \right\}. \]  
(7)

\[ q_t \left\{ \begin{array}{l} [1 - (1 - \alpha)\pi] u' \left( \frac{n_1^t M_1^t}{\psi_1^t} \right) n_1^t \\ + (1 - \alpha)\pi u' \left( \frac{n_2^t M_2^t}{\psi_1^t} \right) n_2^t \end{array} \right\} \]  
(8)

Then, (6) and (7) solve for \( \{n_1^t, n_2^t\}_{t=0}^\infty \) given \( \{M_1^t, M_2^t, \psi_1^t, \psi_2^t\}_{t=0}^\infty \), and \( \{\tau_t\}_{t=0}^\infty \), and (8) then solves for \( \{q_t\}_{t=0}^\infty \).

3 Equilibrium Solution with Constant Money Growth

Let the money growth factor be a constant \( \mu \), and engineer a lump sum transfer to connected households at the first date such that the ratio of money balances held by connected households to money balances held by unconnected households is given by
\[ \delta = \frac{\mu - 1 + \alpha\pi}{\alpha\pi}. \]

Then, this distribution of money balances will persist forever in an equilibrium where labor supplies are constant forever, with labor supplies for connected and unconnected households, respectively, \( n_1 \) and \( n_2 \), determined in a manner analogous to what was done in the previous section, by
\[ \frac{v'(n_1^t)}{\psi_1^t} = \beta \left\{ \begin{array}{l} [1 - (1 - \alpha)\pi] u' \left( \frac{n_1^t M_1^t}{\psi_1^t} \right) n_1^t \\ + (1 - \alpha)\pi u' \left( \frac{n_2^t M_2^t}{\psi_1^t} \right) n_2^t \end{array} \right\}, \]  

\[ \frac{v'(n_2^t)}{\psi_2^t} = \beta \left\{ \begin{array}{l} \alpha\pi u' \left( \frac{n_1^t M_1^t}{\psi_1^t} \right) n_2^t \\ + (1 - \alpha)\pi u' \left( \frac{n_2^t M_2^t}{\psi_2^t} \right) n_2^t \end{array} \right\}. \]
and
\[
v'(n_2)n_2 = \frac{\beta}{\mu} \left\{ \frac{\alpha \pi u'}{\delta + (1 - \alpha)\pi(1 - \delta)} \left( \frac{n_1}{n_1 + (1 - \alpha)\pi(1 - \delta)} \right) n_1 + (1 - \alpha\pi)u' \left( \frac{n_2}{1 + \alpha\pi(\delta - 1)} \right) n_2 \right\},
\]

Further, consumption allocations are
\[
\begin{align*}
c_{1t}^{11} &= \frac{n_1 \delta}{\delta + (1 - \alpha)\pi(1 - \delta)}, \\
c_{1t}^{12} &= \frac{n_2 \delta}{1 + \alpha\pi(\delta - 1)}, \\
c_{2t}^{22} &= \frac{n_2}{1 + \alpha\pi(\delta - 1)}, \\
c_{2t}^{21} &= \frac{n_1}{\delta + (1 - \alpha)\pi(1 - \delta)}.
\end{align*}
\]

4 Proof of Proposition 1

The following equations solve for \( \{n_1^t, n_2^t\}_{t=0}^{\infty} \) with \( \{M_1^t, M_2^t, \psi_1^t, \psi_2^t\}_{t=0}^{\infty} \) given
\[
\begin{align*}
v'(n_1^t)n_1^t &= \frac{\beta}{\psi_t^1} \left\{ [1 - (1 - \alpha)\pi]u' \left( \frac{n_{1t+1} M_{1t+1}^1}{\psi_{t+1}^1} \right) \frac{n_{1t+1}^1}{\psi_{t+1}^1} + (1 - \alpha)\pi u' \left( \frac{n_{2t+1} M_{2t+1}^1}{\psi_{t+1}^2} \right) \frac{n_{2t+1}^1}{\psi_{t+1}^2} \right\}, \\
v'(n_2^t)n_2^t &= \frac{\beta}{\psi_t^2} \left\{ \alpha\pi u' \left( \frac{n_{1t+1} M_{1t+1}^2}{\psi_{t+1}^1} \right) \frac{n_{1t+1}^2}{\psi_{t+1}^2} + (1 - \alpha\pi)u' \left( \frac{n_{2t+1} M_{2t+1}^2}{\psi_{t+1}^2} \right) \frac{n_{2t+1}^2}{\psi_{t+1}^2} \right\},
\end{align*}
\]

for \( t = 0, 1, 2, ... \). Then, approximating the solution to \( n_1^0 \) and \( n_2^0 \) by assuming that the effects of the money supply increase on \( n_i^t \) for \( t > 0, i = 1, 2 \), are negligible, with \( n_i^t = n^* \) for \( t > 0, i = 1, 2 \), from equations (9) and (10) we get
\[
\begin{align*}
V(n_1^0) &= \beta \left\{ [1 - (1 - \alpha)\pi]U \left( \frac{n^* M_1^1}{M_2^1} \right) + (1 - \alpha)\pi U \left( \frac{n^* M_1^1}{M_2^1} \right) \right\}, \\
V(n_2^0) &= \beta \left\{ \alpha\pi U \left( \frac{n^* M_2^2}{M_2^1} \right) + (1 - \alpha\pi)U \left( \frac{n^* M_2^2}{M_2^1} \right) \right\},
\end{align*}
\]

where
\[
U(c) = u'(c)c
\]
and
\[
V(n) = v'(n)n
\]

Now, note that in the baseline case we have \( \frac{M_1^i}{M_2^i} = 1 \) for \( i, j = 1, 2 \), so from (11) and (12) we have \( n_1^0 = n_2^0 = n^* \). Then, using second-order Taylor series
expansions of the terms on the right-hand side of (11) and (12) around \( n^* \), we obtain

\[
V(n_0^i) \approx \beta U(n^*) + \beta U'(n^*) n^*_i \rho_i + \beta U''(n^*) (n^*)^2 \sigma_i
\]

for \( i = 1, 2 \), where

\[
\rho_1 = \left[ \frac{1 - (1 - \alpha)\pi}{M_1^2} \right] + (1 - \alpha)\pi \left[ \frac{M_1^1}{M_2^2} \right] - 1,
\]

\[
\rho_2 = \alpha\pi \left[ \frac{M_1^2}{M_2^2} \right] + (1 - \alpha)\pi \left[ \frac{M_1^1}{M_2^2} \right] - 1,
\]

\[
\sigma_1 = \left[ \frac{1 - (1 - \alpha)\pi}{M_1^2} \right] + (1 - \alpha)\pi \left[ \frac{M_1^1}{M_2^2} \right] - 1,
\]

\[
\sigma_2 = \alpha\pi \left[ \frac{M_1^2}{M_2^2} \right] + (1 - \alpha)\pi \left[ \frac{M_1^1}{M_2^2} \right] - 1.
\]

### 5 Proof of Proposition 3

If each household receives a fixed endowment each period, then the solution for the price of the nominal bond in period 0 is

\[
q_0 = \beta \left\{ \frac{[1 - (1 - \alpha)\pi] u' \left( \frac{\gamma M_1^1}{M_2^2} \right) \pi_2}{\rho_2} + (1 - \alpha)\pi u' \left( \frac{\gamma M_1^1}{M_2^2} \right) \pi_2 \right\}
\]

\[
\left\{ \frac{1 - (1 - \alpha)\pi} {\rho_1} \right\}
\]

(14)

Now, note that the right-hand side of (14) is a function of \( \Delta \), the money supply increase. Taking a first-order Taylor series approximation to the right-hand side of (14) around \( \Delta = 0 \), we obtain

\[
q_0 \approx \beta + \frac{\beta(1 - \alpha)\pi \Delta}{\alpha M} \left\{ \gamma(y) \left[ 1 - (1 - \pi)^2 \right] + (1 - \pi)^2 \right\},
\]

(15)

where \( \gamma(y) = -\frac{\delta u''(y)}{\beta (1 - \pi)} \) is the coefficient of relative risk aversion.

### 6 Proof of Proposition 4

We can derive the effect of a change in the money growth factor \( \mu \) on \( n_1 \) and \( n_2 \), at least for \( \mu = 1 \). That is, totally differentiating

\[
v'(n_1) n_1 = \frac{\beta}{\mu} \left\{ \frac{1 - (1 - \alpha)\pi}{\rho_1} \right\} u' \left( \frac{\gamma M_1^1}{\pi_2} \right) \frac{n_1}{\rho_2}
\]

(16)

and

\[
v'(n_2) n_2 = \frac{\beta}{\mu} \left\{ \frac{\alpha\pi u' \left( \frac{n_1}{\gamma (1 - \alpha)\pi (1 - \pi)} \right) \pi_2}{\rho_1} + (1 - \alpha\pi) u' \left( \frac{n_2}{\gamma (1 - \alpha)\pi (1 - \pi)} \right) \pi_2 \right\},
\]

(17)
using
\[ \delta = \frac{\mu - 1 + \alpha \pi}{\alpha \pi}. \]  
(18)
and evaluating derivatives at \( \mu = 1 \), we obtain
\[ \frac{dn_1}{d\mu} = \frac{\beta}{\nabla} \left( \left( v'' n^* - \beta u'' n^* \right) \{ -u' + \frac{1}{\alpha} \left[ (2 - \pi) n^* u'' + u' \right] \} \right), \]
(19)
\[ \frac{dn_2}{d\mu} = \frac{\beta}{\nabla} \left( \left( v'' n^* - \beta u'' n^* \right) \{ -u' - \left[ (2 - \pi) n^* u'' + u' \right] \} - \beta \pi \left( u'' n^* + u' \right) \right), \]
(20)
where
\[ \nabla = (v'' - \beta u'') [v'' n^* + \beta \pi u' - \beta (1 - \pi) u'' n^* - \beta (1 - \pi) u'' n^*] > 0. \]
In general, we cannot sign \( \frac{dn_1}{d\mu} \) and \( \frac{dn_2}{d\mu} \).

7 Proof of Proposition 5

Use the equilibrium solution in the constant money growth case above to substitute in the welfare function,
\[ W(\mu) = \left\{ \begin{array}{l}
\alpha [1 - (1 - \alpha)\pi u(c_1^{11}) + \alpha (1 - \alpha)\pi u(c_2^{11})] + (1 - \alpha) [1 - (1 - \alpha)\pi u(c_1^{21}) + (1 - \alpha)\alpha \pi u(c_2^{21})]\\
- \alpha v(n_1) - (1 - \alpha) v(n_2)
\end{array} \right\}, \]
(21)
then differentiate and evaluate the derivative for \( \mu = 1 \), to obtain
\[ W'(1) = A + (1 - \beta) u'(n*) \left[ \frac{\alpha}{d\mu} \frac{dn_1}{d\mu} + (1 - \alpha) \frac{dn_1}{d\mu} \right], \]
where \( A \) is the effect of a change in \( \mu \) on welfare caused by the increase in consumption risk arising from the redistribution of consumption goods among agents. The remaining portion of the change in welfare is the net effect on welfare of the change in labor supply resulting from a change in \( \mu \). It is straightforward to show that \( A = 0 \), that is since consumption is equal across agents when \( \mu = 1 \), the first-order effect of a change in \( \mu \) on consumption risk is nil. Therefore, the net effect on welfare when \( \mu = 1 \) is determined by the effect on aggregate output, which in the paper is shown to be negative. Therefore, \( W'(1) < 0 \), and a small reduction in the money growth rate from zero will increase welfare.

8 Stochastic Money Supply

For notational convenience, use primes to denote variables dated \( t + 1 \), with \( \delta \) (\( \delta' \)) denoting the ratio of money holdings of connected and unconnected households in period \( t \) (\( t + 1 \)). As well \( \mu \) (\( \mu' \)) denotes the gross growth rate in the
aggregate money stock in period \( t \) \((t+1)\). Then, under the maintained assumption that cash-in-advance constraints bind in all states of the world, from (18) \( \delta' \) depends only on \( \mu' \), and \( \delta \). Now, if \( \mu = \mu(\delta, \epsilon) \), where \( \epsilon \) is a first-order Markov process, then the state vector is \((\epsilon, \delta)\) and we can look for a recursive competitive equilibrium where quantities and nominal interest rates depend only on the state.

The distribution of money balances across the population evolves according to

\[
\delta' = \frac{\alpha[\mu' - (1 - \alpha)\pi]\delta + (1 - \alpha)(\mu' - 1 + \alpha\pi)}{\alpha^2\pi\delta + \alpha(1 - \alpha\pi)}.
\]

Then, solving for an equilibrium in a manner analogous to what we did in the deterministic case, we can solve for labor supplies of connected and unconnected households, \( n_1 \) and \( n_2 \), which are implicitly a function of the state \((\epsilon, \delta)\), using

\[
v'(n_1)n_1 = \beta \left\{ \frac{[1 - (1 - \alpha)\pi]\delta + (1 - \alpha)\pi}{\alpha\pi\delta + 1 - \alpha\pi} \right\}
\times \mathbb{E} \left\{ \frac{[1 - (1 - \alpha)\pi]\mu'\left(\frac{n_1'\delta'}{\alpha^2\pi\delta' + (1 - \alpha\pi)}\right) + \frac{n_1'}{\alpha^2\pi\delta' + (1 - \alpha\pi)}}{\alpha\pi\delta' + 1 - \alpha\pi} \right\},
\]

and

\[
v'(n_2)n_2 = \beta \mathbb{E} \left\{ \frac{\alpha\pi\mu'\left(\frac{n_2'\delta'}{\alpha^2\pi\delta' + (1 - \alpha\pi)}\right) + \frac{n_2'}{\alpha^2\pi\delta' + (1 - \alpha\pi)}}{\alpha\pi\delta' + 1 - \alpha\pi} \right\},
\]