Notes on Search

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1 Search and Unemployment: One-Sided Search

Unemployment is measured as the number of persons actively seeking work. Clearly, there is no counterpart to this concept in standard representative agent neoclassical growth models. If we want to understand the behavior of the labor market, explain why unemployment fluctuates and how it is correlated with other key macroeconomic aggregates, and evaluate the efficacy of policies affecting the labor market, we need another set of models. These models need heterogeneity, as we want to study equilibria where agents engage in different activities, i.e. job search, employment, and possibly leisure (not in the labor force). Further, there must be frictions which imply that it takes time for an agent to transit between unemployment and employment. Search models have these characteristics.

Some early approaches to search and unemployment are in McCall (1970) and Phelps et. al. (1970). These are models of "one-sided" search, which are partial equilibrium in nature. Unemployed workers face a distribution of wage offers which is assumed to be fixed. Later, Mortensen and Pissarides developed two-sided search models (for a summary see Pissarides 1990) in which workers and firms match in general equilibrium, and wages are endogenous.

1.1 A One-Sided Search Model

Suppose a continuum of agents with unit mass, each having preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t c_t,$$

where $0 < \beta < 1$. Let $\beta = \frac{1}{1+r}$, where r is the discount rate. Note that we assume that there is no disutility from labor effort on the job, or from effort in searching for a job. There are many different jobs in this economy, which differ according to the wage, w, which the worker receives. From the point of view of an unemployed agent, the distribution of wage offers she can receive in any period is given by the probability distribution function $F(w)$, which has associated with it a probability density function $f(w)$. Assume that $w \in [0, \bar{w}]$, i.e. the set $[0, \bar{w}]$ is the support of the distribution. If an agent is employed receiving wage w (assume that each job requires the input of one unit of labor each period), then her consumption is also w, as we assume that the worker has no opportunities to save. At the end of the period, there is a probability δ that an employed worker will become unemployed. The parameter δ is referred to as the separation rate. An unemployed worker receives an unemployment benefit, b, from the government at the beginning of the period, and then receives a wage offer that she may accept or decline. Assume that $b < \bar{w}$, so that at least some job offers have higher compensation than the unemployment insurance benefit.

Let V_u and $V_e(w)$ denote, respectively, the value of being unemployed and the value of being employed at wage w, as of the end of the period. These values are determined by two Bellman equations:

$$V_u = \beta \left\{ b + \int_0^{\bar{w}} \max\left[V_e(w), V_u\right] f(w)dw \right\} \tag{1}$$

$$V_e(w) = \beta\left[w + \delta V_u + (1-\delta)V_e(w)\right] \tag{2}$$

In (1), the unemployed agent receives the unemployment insurance benefit, b, at the beginning of the period, consumes it, and then receives a wage offer from the distribution $F(w)$. The wage offer is accepted if $V_e(w) \geq V_u$ and declined otherwise. The integral in (1) is the expected utility of sampling from the wage distribution.

In (2), the employed agent receives the wage, w, consumes it, and then either suffers a separation or will continue to work at the wage w next period. Note that an employed agent will choose to remain employed if she does not experience a separation, because $V_e(w) \geq V_u$, otherwise she would not have accepted the job in the first place.

In search models, a useful simplification of the Bellman equations is obtained as follows. For (1), divide both sides by β, substitute $\beta = \frac{1}{1+r}$, and subtract V_u from both sides to obtain

$$rV_u = b + \int_0^{\bar{w}} \max\left[V_e(w) - V_u, 0\right] f(w)dw. \tag{3}$$

On the right-hand side of (3) is the flow return when unemployed plus the expected net increase in expected utility from the unemployed state. Similarly, (2) can be simplified to obtain

$$rV_e(w) = w + \delta[V_u - V_e(w)] \tag{4}$$

We now want to determine what wage offers an agent will accept when unemployed. From (4), we obtain

$$V_e(w) = \frac{w + \delta V_u}{r + \delta}. \tag{5}$$

Therefore, $V_e(w)$ is a strictly increasing linear function of w. Thus, there is some w^* such that $V_e(w) \geq V_u$ for $w \geq w^*$, and $V_e(w) \leq V_u$ for $w \leq w^*$. The value w^* is denoted the *reservation wage.* That is, an unemployed agent will accept any wage offer of w^* or more, and decline anything else. The reservation wage satisfies $V_e(w^*) = V_u$, so from (5), we have

$$V_u = \frac{w^*}{r}. \tag{6}$$

Then, if we substitute for V_u in equation (3) using (6) and for $V_e(w)$ using (5), we get

$$w^* = b + \int_0^{\bar{w}} \max\left[\frac{w - w^*}{r + \delta}, 0\right] f(w)dw,$$

or, simplifying,

$$w^* = b + \frac{1}{r+\delta}\int_{w^*}^{\bar{w}}(w - w^*)f(w)dw,$$

and simplifying further,

$$w^* = b + \frac{1}{r+\delta}\left\{\int_{w^*}^{\bar{w}} wf(w)dw - w^*[1 - F(w^*)]\right\}.$$

Next integrate by parts to obtain

$$w^* = b + \frac{1}{r+\delta}\left\{\bar{w} - w^*F(w^*) - \int_{w^*}^{\bar{w}} F(w)dw - w^*[1 - F(w^*)]\right\},$$

and simplify again to get

$$w^* = b + \frac{1}{r+\delta}\int_{w^*}^{\bar{w}}[1 - F(w)]dw. \tag{7}$$

Equation (7) solves for the reservation wage w^*. Note that the left-hand side of this equation is a strictly increasing and continuous function of w^*, while the right-hand side is a decreasing and continuous function of w^*. For $w^* = 0$, the right-hand side of the equation exceeds the left-hand side, and for $w = \bar{w}$ the left-hand side exceeds the right. Therefore, a solution for w^* exists, and it is unique. We depict the determination of the reservation wage in Figure 1.1, where

$$A(w^*) = \frac{1}{r+\delta}\int_{w^*}^{\bar{w}}[1 - F(w)]dw.$$

In the figure, the reservation wage is w_1^*. Note from the figure that we must have $w_1^* > b$. That is, while it is intuitively clear that an unemployed worker would never accept a job offering a wage smaller than the unemployment insurance benefit, he or she would also not accept a wage offer that exceeds b by a small amount. This is because an unemployed worker is willing to turn down such an offer and continue to collect b, hoping to receive a wage offer that is much higher in the future.

1.2 Comparative Statics

It is now straightforward to use equation (7) to determine how changes in agents' preferences and in the environment affect the reservation wage w^*. First, consider a change in the unemployment insurance benefit, b. Totally differentiating equation (7) and solving gives

$$\frac{dw^*}{db} = \frac{r+\delta}{r+\delta+1-F(w^*)} > 0.$$

Therefore, as shown in Figure 1.2, the reservation wage increases with an increase in the unemployment insurance benefit. This occurs because an increase in b reduces the cost of search while unemployed. An unemployed worker therefore becomes more picky concerning the jobs he or she will accept.

Next, note from equation (7) that r and δ will affect the determination of the reservation wage in exactly the same way, so we can kill two birds with one stone, totally differentiating (7) in a similar fashion to what we did for a change in b to get

$$\frac{dw^*}{dr} = \frac{dw^*}{d\delta} = \frac{-1}{(r+\delta)\left[r+\delta+1-F(w^*)\right]}\int_{w^*}^{\bar{w}}[1-F(w)]dw < 0.$$

Therefore, an increase in either r or δ reduces the reservation wage. If r increases, then agents discount future payoffs at a higher rate, and therefore are less willing to wait for a better wage offer in the future. They become less picky and reduce their reservation wage. If the separation rate δ increases, this will reduce the difference between the value of being employed and the value of being unemployed (given the reservation wage), which from (5) and (6) is

$$V_e(w) - V_u = \frac{w-w^*}{r+\delta}.$$

This effect occurs because higher δ implies that the expected lifetime of a job is lower. The effect of all jobs being less attractive, perhaps counterintuitively, is that unemployed workers become less picky about the jobs they will accept, because it is not so tempting to hold out for a better job that will now tend to dissolve more quickly.

Other experiments that we could consider involve changes in the distribution of wage offers $F(w)$. What equation (7) tells us is that the wage offer distribution matters for the determination of the reservation wage w^* in terms of how it affects

$$G(F) = \int_{w^*}^{\bar{w}}[1-F(w)]dw.$$

That is, if a change in F increases $G(F)$, then it has a qualitative effect on the reservation wage identical to the effect of an increase in b, as in Figure 1.2. That is, w^* increases. This would be the effect if, for example, there were a first-order-stochastic-dominance shift in $F(w)$, whereby $F(w)$ decreases for all $w \in (0, \bar{w})$. Thus, if the wage distribution improves in the sense of first-order stochastic dominance, then w^* must increase because the expected gain from turning down a wage offer and waiting for a better one increases. Note also that $G(F)$ can increase if the dispersion in the distribution $F(w)$ increases in particular ways. For example, if dispersion increases in such a way that the probability mass to the right of the initial w^* remains the same (i.e. $F(w^*)$ does not change for the initial w^*), then $G(F)$ increases and w^* must increase.

1.3 Employment and Unemployment

Now that we have determined the behavior of individual agents, as summarized by how w^* is determined, we can say something about the behavior of aggregate employment and unemployment. Now, let u_t denote the fraction of agents who are unemployed in period t. The flow of agents into employment is just the fraction of unemployed agents multiplied by the probability that an individual agent transits from unemployment to employment, $u_t\left[1-F(w^*)\right]$. Further, the flow of agents out of employment to unemployment is the number of separations $(1-u_t)\delta$. Therefore, the law of motion for γ_t is

$$\begin{aligned} u_{t+1} &= u_t - u_t\left[1-F(w^*)\right] + (1-u_t)\delta \\ &= u_t[F(w^*)-\delta] + \delta. \end{aligned} \tag{8}$$

Since $\mid F(w^*)-\delta \mid< 1$, u_t converges to a constant, u, which is determined by setting $u_{t+1} = u_t = u$ in (8) and solving to get

$$u = \frac{\delta}{\delta + 1 - F(w^*)}. \tag{9}$$

Therefore, the number of unemployed increases as the separation rate increases, and as the reservation wage increases (though note that the reservation wage also depends on the separation rate). That is, a higher separation rate increases the flow from employment to unemployment, increasing the unemployment rate, and a higher reservation wage reduces the job-finding rate, $1-F(w^*)$, thus reducing the flow from unemployment to employment and increasing the unemployment rate.

We can conclude, from our analysis of what affects the reservation wage w^*, that an increase in b or a decrease in r, which each increases the reservation wage, will also increase the unemployment rate, from (9). An increase in the separation rate δ has the direct effect of increasing the unemployment rate, but it also will reduce the reservation wage, which will reduce the unemployment rate. The net effect is ambiguous. Similarly, a first-order stochastic dominance shift in the wage offer distribution $F(w)$ has the effect of increasing the reservation wage and therefore reducing the unemployment rate, but since $F(w)$ falls for each $w \in (0, \bar{w})$, the net effect on $F(w^*)$ is ambiguous. The unemployment rate could increase or decrease. However, if $F(w)$ changes in such a way that dispersion increases while holding $F(w^*)$ constant for the initial w^*, then w^* increases, $F(w^*)$ increases, and the increase in dispersion increases the unemployment rate.

1.4 An Example

Suppose that there are only two possible wage offers. An unemployed agent receives a wage offer of $\bar{w}$ with probability π and an offer of zero with probability $1-\pi$, where $0 < \pi < 1$. Suppose first that $0 < b < \bar{w}$. Here, in contrast to the general case above, the agent knows that when she receives the high wage offer, there is no potentially higher offer that she foregoes by accepting, so high wage offers are always accepted. Low wage offers are not accepted because collecting unemployment benefits is always preferable, and the agent cannot search on the job. Letting V_e denote the value of employment at wage $\bar{w}$, the Bellman equations can then be written as

$$rV_u = b + \pi(V_e - V_u),$$

$$rV_e = \bar{w} + \delta(V_u - V_e),$$

and we can solve these two equations in the unknowns V_e and V_u to obtain

$$V_e = \frac{(r+\pi)\bar{w} + \delta b}{r(r+\delta+\pi)},$$

$$V_u = \frac{\pi\bar{w} + (r+\delta)b}{r(r+\delta+\pi)}.$$

Note that

$$V_e - V_u = \frac{\bar{w} - b}{r+\delta+\pi}$$

depends critically on the difference between $\bar{w}$ and b, and on the discount rate, r. The number of unemployed agents in the steady state is given by

$$u = \frac{\delta}{\delta+\pi},$$

so that the number unemployed decreases as π increases, and rises as δ increases.

Now for any $b > \bar{w}$, clearly we will have $\gamma = 0$, as no offers of employment will be accepted, due to the fact that collecting unemployment insurance dominates all alternatives. However, if $b = \bar{w}$, then an unemployed agent will be indifferent between accepting and declining a high wage offer. It will then be optimal for her to follow a mixed strategy, whereby she accepts a high wage offer with probability η. Then, the number of employed agents in the steady state is

$$u = \frac{\delta}{\delta+\eta\pi},$$

which is decreasing in η. This is a rather stark example where changes in the UI benefit have no effect before some threshold level, but increasing benefits above this level causes everyone to turn down all job offers.

1.5 Discussion

The partial equilibrium approach above has neglected some important factors, in particular the fact that, if job vacancies are posted by firms, then the wage offer distribution will be endogenous - it is affected by the rate at which the unemployed accept different jobs, and by what types of jobs are posted by firms. In addition, we did not take account of the fact that the government must somehow finance the payment of unemployment insurance benefits. A simple financing scheme in general equilibrium would be to have UI benefits funded from lump-sum taxes on employed agents.

1.6 References

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2 A Two-Sided Search Model: Mortensen-Pissarides

For many macroeconomic issues, we want general equilibrium search models of unemployment in which we can determine wages endogenously and seriously address the effects of policy. Versions of two-sided search and matching models, developed first in the late 1970s, have been used extensively in labor economics and macro. For further references see Mortensen (1985), Pissarides (1990), and Rogerson, Shimer, and Wright (2005). What I have done here borrows heavily from the latter survey, though I work here exclusively in discrete time.

2.1 The Model

There is a continuum of workers with unit mass, each of whom has preferences given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t c_t,$$

where c_t is consumption and $r > 0$. There is also an infinite mass of firms, with each firm having preferences given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t (\pi_t - x_t),$$

where π_t denotes the firm's profits, which are consumed by the firm, and x_t denotes any disutility from posting a vacancy during period t. Goods are perishable, and savings is assumed to be zero for each agent in each period.

Let u_t denote the mass of workers who are unemployed each period, with $1-u_t$ being the mass of workers who are matched with firms, producing output, and therefore employed. As well, v_t is the mass of firms which post vacancies in period t. Each period, there are matches between unemployed workers and firms posting vacancies, with m_t denoting the mass of matches according to

$$m_t = m(u_t, v_t),$$

where $m(\cdot,\cdot)$ is the *matching function.* Assume that $m(\cdot,\cdot)$ is continuous, increasing in both arguments, concave, homogeneous of degree 1, and that $m(0,v) = m(u,0) = 0$ for all $v, u \geq 0$. The probability with which an individual unemployed worker is matched with a firm posting a vacancy in period t is given by $\frac{m(u_t,v_t)}{u_t} = m(1, \frac{v_t}{u_t})$ given that the matching function satisfies homogeneity of degree 1. Similarly, the probability that an individual firm posting a vacancy is matched with a worker is $\frac{m(u_t,v_t)}{v_t} = m(\frac{u_t}{v_t}, 1)$. For convenience, define $\theta_t \equiv \frac{v_t}{u_t}$, where θ_t is a measure of labor market tightness in period t, in that an increase in θ_t increases the job-finding probability for an unemployed worker, and lowers the probability that a firm can fill a job. Assume that

$$\lim_{\theta \to 0} m\left(\frac{1}{\theta}, 1\right) = \lim_{\theta \to \infty} m(1,\theta) = 1.$$

Each firm has a technology for producing output. With this technology, y units of output can be produced with one unit of labor input each period, and zero units of output for any other quantity of labor input. Each worker has one unit of time available each period. When a worker and firm meet, and agree to a contract, they can then jointly produce y units of output until they become separated. Separation occurs each period with probability δ. While unemployed, a worker receives unemployment insurance compensation of b each period (note that, as in the one-sided model, we don't account for the financing of b by the government). A firm posting a vacancy incurs a cost in terms of utility of k each period the vacancy is posted. Any firm not posting a vacancy and not matched with a worker receives zero utility.

2.2 Bargaining

We will confine attention to steady state equilibria where $u_t = u$ and $v_t = v$ for all t. When a worker and firm meet, they will negotiate a wage w, which is the payment that will be made to the worker in each period until the firm and worker are separated. Let $W(w)$ denote the value of the match to a worker if the wage is w, and let $J(y-w)$ denote the value of the match to the firm. As well, let U denote the value to the worker of remaining unemployed, and V the value to the firm of posting a vacancy. Here, all values are defined to be as of the end of the period. The worker and the firm can only come to an agreement if $W(w)-U \geq 0$ and $J(y-w)-V \geq 0$ for some w, where $W(w)-U$ denotes the surplus from the match for the worker, and $J(y-w)-V$ denotes the surplus from the match for the firm. The total surplus is the sum of these two quantities, or $W(w)+J(y-w)-U-V$. A tractable approach to the determination of the equilibrium wage is to suppose that the firm and worker engage in Nash bargaining, so that

$$w = \arg\max_{w'} \left[W(w')-U\right]^{\alpha} \left[J(y-w')-V\right]^{1-\alpha}$$

subject to

$$W(w')-U \geq 0,$$
$$J(y-w')-V \geq 0.$$

where α is a parameter which is a measure of the worker's bargaining power, with $0 \leq \alpha \leq 1$. Note that the above optimization problem is not a problem solved by any individual agent - instead the solution to this problem describes the outcome of bargaining between the worker and the firm.

Ignoring the constraints in the above optimization problem for now, the first-order condition for a maximum simplifies to give

$$\alpha W'(w)[J(y-w)-V] - (1-\alpha)J'(y-w)[W(w)-U] = 0. \tag{10}$$

For a worker, the value of being employed at wage w, as of the end of the period, is given by

$$W(w) = \frac{1}{1+r}\left[w + (1-\delta)W(w) + \delta U\right],$$

and the value of a match for a firm, given the wage w, is

$$J(y-w) = \frac{1}{1+r}\left[y - w + (1-\delta)J(y-w) + \delta V\right].$$

Simplifying these two Bellman equations, just as we did for the one-sided search model, gives, respectively,

$$rW(w) = w + \delta[U - W(w)] \tag{11}$$

and

$$rJ(y-w) = y - w + \delta[V - J(y-w)]. \tag{12}$$

Therefore, from (11) and (12), we obtain, respectively,

$$W(w) = \frac{w + \delta U}{r+\delta},$$

and

$$J(y-w) = \frac{y - w + \delta V}{r+\delta},$$

and so $W'(w) = J'(y-w) = \frac{1}{r+\delta}$. Note here that U and V will in general depend on the wages paid by other firms, but this is independent of the wage that is being negotiated in the particular labor contract between an individual worker and an individual firm. Therefore, equation (10) simplifies to

$$\alpha[J(y-w) - V] - (1-\alpha)[W(w) - U] = 0. \tag{13}$$

2.3 Equilibrium

Next, we need Bellman equations determining values for an unemployed worker and for a firm posting a vacancy. Since in equilibrium all jobs will pay the same wage, we will let W denote the equilibrium value of being employed for a worker and J the value of a match for a firm. Further, suppose that any meeting between a firm and worker results in a successful match. Then, U and V are determined, respectively, by

$$U = \frac{1}{1+r}\left\{b + m(1,\theta)W + [1 - m(1,\theta)]U\right\},$$

and

$$V = \frac{1}{1+r}\left\{-k + m\left(\frac{1}{\theta},1\right)J + \left[1 - m\left(\frac{1}{\theta},1\right)\right]V\right\},$$

or simplifying,

$$rU = b + m(1,\theta)(W - U), \tag{14}$$

$$rV = -k + m\left(\frac{1}{\theta},1\right)(J - V). \tag{15}$$

The final detail we need in the model is the analog of a zero-profit condition for firms. That is, in a steady state equilibrium, firms have to be indifferent between their alternative opportunity, which yields zero value, and posting a vacancy. That is

$$V = 0. \tag{16}$$

Let S denote the total surplus from a match for a firm and a worker, where

$$S = W + J - U - V = W + J - U \tag{17}$$

Then, equation (13) gives

$$W - U = \alpha S, \tag{18}$$

that is Nash bargaining implies here that the worker gets a constant fraction α of the total surplus, determined by the worker's bargaining power. Therefore, it follows that

$$J - V = (1 - \alpha)S. \tag{19}$$

Next, (11), (12), and (14) imply, subtracting (14) from (11) plus (12),

$$r(W + J - U - V) = y - b - k + \delta(U - W - J) - m(1, \theta)(W - U) \tag{20}$$

Then from (17), (18), and (19), we can simplify (20) to get

$$S = \frac{y - b}{r + \delta + m(1, \theta)\alpha}, \tag{21}$$

and from (15), (19), and given $V = 0$, we get

$$S = \frac{k}{(1 - \alpha)m\left(\frac{1}{\theta}, 1\right)}. \tag{22}$$

Equations (21) and (22) solve for S and θ. Then, we can solve for all other endogenous variables. From (12), given S the wage is determined by

$$w = y - (r + \delta)(1 - \alpha)S, \tag{23}$$

then given S and w, (11) gives

$$W = \frac{w + \delta\alpha S}{r}, \tag{24}$$

and since $W - U = \alpha S$, then

$$U = \frac{w + (\delta - r)\alpha S}{r}. \tag{25}$$

In the steady state, the flow of workers from unemployment to employment is $um(1, \theta)$, while the flow of workers from employment to unemployment is $(1 - u)\delta$. In a steady state, these flows are equal, which implies that, given θ, u is given by

$$u = \frac{\delta}{m(1, \theta) + \delta}, \tag{26}$$

and given the definition of θ, we then have

$$v = u\theta = \frac{\delta\theta}{m(1,\theta)+\delta}. \tag{27}$$

Now, let $F(\theta)$ denote the right-hand side of (21) and $G(\theta)$ the right-hand side of (22). The functions $F(\cdot)$ and $G(\cdot)$ are continuous with $F'(\theta) < 0$ and $G'(\theta) > 0$, $F(0) = \frac{y-b}{r+\delta}$, $G(0) = \frac{k}{1-\alpha}$, $F(\infty) = \frac{y-b}{r+\delta+\alpha}$, and $G(\infty) = \infty$. Therefore, an equilibrium exists if and only if

$$k < \frac{(1-\alpha)(y-b)}{r+\delta},$$

that is, if and only if the cost of posting a vacancy is sufficiently small. If this condition holds, then the equilibrium is unique, as in Figure 2.1, where $S = S^*$ and $\theta = \theta^*$ in equilibrium, and we will have

$$\frac{y-b}{r+\delta+\alpha} < S^* < \frac{y-b}{r+\delta},$$

so $S > 0$ in equilibrium, which in turn implies that both a matched worker and a matched firm earn positive surplus. Therefore, our conjecture that each meeting between a firm and a worker results in a successful match is correct.

2.4 Experiments

Consider first a change in y, interpreted as an increase in aggregate productivity. In Figure 2.1, an increase in y will result in an increase in S and an increase in θ. From (26), unemployment must then fall. To determine the effect on vacancies, differentiate (27) with respect to θ to get

$$\frac{dv}{d\theta} = \frac{m(1,\theta)+\delta-m_2(1,\theta)\theta}{\left[m(1,\theta)+\delta\right]^2} = \frac{m_1(1,\theta)+\delta}{\left[m(1,\theta)+\delta\right]^2} > 0,$$

which uses the homogeneity-of-degree-one property of the matching function. It can also be shown that the wage w increases. The mechanism at work here is that an increase in y will tend to increase the total surplus from a match, making posting vacancies more attractive for firms, so that v and θ increase. This increases the job-finding rate for unemployed workers, and the unemployment rate falls. The increase in productivity makes unemployment and vacancies move in opposite directions. Though we are looking at a steady state equilibrium, this mechanism works similarly in stochastic versions of two-sided search models, and will tend to yield a negative correlation between u and v, referred to as a *Beveridge curve.*

A decrease in b has the opposite effects of an increase in y. An increase in unemployment insurance compensation acts to reduce total surplus in a match and therefore makes posting vacancies less attractive for firms, so that v and θ fall. This reduces the job-finding rate and u rises. Note from equation (23) that the wage rises, since unemployment is more attractive for workers, and firms therefore have to pay higher wages to make employment sufficiently attractive for workers.

Finally, consider an increase in the separation rate δ. From Figure 2.1, this has the effect of reducing both S and θ. Unemployment u must rise, both because of the direct effect of δ on u, and because of the decrease in θ which reduces the job-finding rate. The decrease in θ causes v to fall, but the direct effect of δ on v is for v to rise, and it is possible for u and v to both rise. As for the case of changes in productivity, the mechanism at work here also transfers to stochastic environments, so that shocks to the separation rate may tend to produce a positive correlation between u and v, which is not observed in the data. Therefore, at least in terms of qualitative features of the data, productivity shocks do a better job than do separation rate shocks in explaining what we observe.

2.5 Discussion

As with the previous one-sided search model, we have left out the details of the financing of unemployment insurance payments, so this is not quite a general equilibrium model. For this model to successfully address problems in business cycle behavior and policy (such as the optimal design of unemployment insurance systems), we also need to be more serious about savings, investment, and capital accumulation.

A fundamental weakness in the standard two-sided matching model is the matching function specification. This is basically a cheap way to capture heterogeneity in the model without specifying it explicitly. That is, workers and firms have difficulty matching in practice because there is heterogeneity on both sides of the market, and because there is private information about worker types and firm types. The matching function is not likely to be immune from the *Lucas critique* in many policy applications. That is, the matching function is not a structural object. We would not expect the function to be invariant to changes in policies. For example, if government labor market policy changes, this will in general cause firms and workers to match at a different rate.

There have been a number of interesting applications of stochastic two-sided search models to business cycle problems. These applications include Andolfatto (1995), Merz (1995), and Shimer (2005).

2.6 References

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3 Monetary Search with Indivisibilities

Traditionally, money has been viewed as having three functions: it is a *medium of exchange,* a *store of value,* and a *unit of account.* Money is a medium of exchange in that it is an object which has a high velocity of circulation; its value is not derived solely from its intrinsic worth, but from its wide acceptability in transactions. It is hard to conceive of money serving its role as a medium of exchange without being a store of value, i.e. money is an asset. Finally, money is a unit of account in that virtually all contracts are denominated in terms of it.

Jevons (1875) provided an early account of a friction which gives rise to the medium-of-exchange role of money. The key elements of Jevons's story are that economic agents are specialized in terms of what they produce and what they consume, and that it is costly to seek out would-be trading partners. For example, suppose a world in which there is a finite number of different goods, and each person produces only one good and wishes to consume some other good. Also suppose that all trade in this economy involves barter, i.e. trades of goods for goods. In order to directly obtain the good she wishes, it is necessary for a particular agent to find someone else who has what she wants, which is a single coincidence of wants. A trade can only take place if that other person also wants what she has, i.e. there is a double coincidence of wants. In the worst possible scenario, there is an *absence of double coincidence of wants,* and no trades of this type can take place. At best, trading will be a random and time-consuming process, and agents will search, on average, a long time for trading partners.

Suppose now that we introduce money into this economy. This money could be a commodity money, which is valued as a consumption good, or it could be fiat money, which is intrinsically useless but difficult or impossible for private agents to produce. If money is accepted by everyone, then trade can be speeded up considerably. Rather than having to satisfy the double coincidence of wants, an agent now only needs to find someone who wants what she has, selling her production for money, and then find an agent who has what she wants, purchasing their consumption good with money. When there is a large number of goods in the economy, two single coincidences on average occur much sooner than one double coincidence.

The above story has elements of search in it, so it is not surprising that the search structure used by labor economists and others could be applied in monetary economics. One of the first models of money and search is that of Jones (1976), but the more recent monetary search literature begins with Kiyotaki and Wright (1989). Kiyotaki and Wright's model involves three types of agents and three types of goods (the simplest possible kind of absence of double coincidence model), and is useful for studying commodity monies, but is not a very tractable model of fiat money. The model we study in this section is related to Kiyotaki and Wright (1993), where symmetry is exploited to obtain a framework where it is convenient to study the welfare effects of introducing fiat money.

3.1 The Model

There is a continuum of agents with unit mass, each having preferences given by

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t [u(c_t) - h_t],$$

where $r > 0$, c_t denotes consumption, $u(\cdot)$ is an increasing function, and h_t is labor supply. There is a continuum of goods, and a given agent can produce only one of the goods in the continuum. An agent gets zero utility from consuming her production good. Each period, agents meet pairwise and at random. For a given agent, the probability that her would-be trading partner can produce a good that she likes to consume is x, and the probability that she can produce what her would-be trading partner wants is also x. There is also a good called money, and a fraction M of the population is endowed with one unit each of this stuff in period 0. All goods are indivisible, being produced and consumed in one-unit quantities. Producing a good requires γ units of labor, where $\gamma > 0$. Privately produced goods cannot be stored - they are perishable. However money, which no private agent can produce, is storable at zero cost. An agent cannot store more than one unit of money at a time. Free disposal is assumed, so it is possible to throw money away. For convenience, let $u^* = u(1)$ denote the utility from consuming a good that the agent likes, and assume that $u(0) = 0$.

At the beginning of period 0, a fraction M of agents is chosen at random and endowed with 1 unit of money each. Then, in period 0 and in each succeeding period, each agent is randomly matched with another agent. Assume that, if an agent is holding money that he or she cannot produce.

3.2 Perfect Memory: An Economy with "Credit"

To highlight the key frictions that give rise to a role for monetary exchange in this economy, we will first assume that there is perfect recordkeeping, or perfect memory. That is, when two agents meet, each knows the other's complete history. An agent's history includes a record of past consumption and production, and who the agent traded with. In this context, also assume that there is limited commitment, in that an agent cannot commit to future actions, and cannot be forced to produce.

We first would like to determine what is optimal in this economy. Think of this as a mechanism design problem, where we determine what is optimal, subject to incentive constraints, from the point of view of a benevolent social planner. As should be clear, assuming perfect memory implies that money is of no use in this economy, so the social planner will have agents throw their money away at the first date. It should also be clear that the optimal allocation, if it is feasible, is one where, when two agents a and b meet, if a likes b's good then b should produce one unit for a, and if b likes a's good then a should produce for b. Expected period utility will then be $x(u^* - \gamma)$ for each agent, and so expected lifetime utility for each agent at the beginning of period 0 is

$$\hat{U} = \frac{(1+r)\,x(u^* - \gamma)}{r}. \tag{28}$$

Now, for this allocation to be feasible, each agent must have the incentive to produce each period. Clearly, an agent will not produce unless the planner imposes some punishment for deviating. The worst punishment the planner can impose on an agent is autarky forever, which yields lifetime utility $U^A = 0$, and this is the optimal punishment, since it will not be imposed in equilibrium. Then, for the allocation to be feasible, it must not be in the interest of an agent in a single-coincidence match (where only one of the agents likes the other's good) to deviate, or

$$-\gamma + \frac{\hat{U}}{1+r} \geq U^A. \tag{29}$$

Similarly, an agent must want to produce in a double-coincidence match, or

$$u^* - \gamma + \frac{\hat{U}}{1+r} \geq U^A. \tag{30}$$

Clearly, if (29) holds, then so does (30). We can rewrite (29) as

$$\gamma \leq \frac{x(u^* - \gamma)}{r}, \tag{31}$$

so the cost of cooperating (the cost of production) must be smaller than the benefit (the expected discounted utility from participating in the mechanism in the future). We will assume that (31) holds.

Now, given the environment, consider equilibria in a dynamic game, where the strategy an agent chooses each period is whether to produce for the agent he or she is matched with. There are many equilibria to this game, but one equilibrium is the one that will reproduce the optimal allocation. This equilibrium is supported by an off-equilibrium punishment whereby, if any agent deviates, then no one will produce for him or her in the future. We can think of this as a credit equilibrium with a centralized credit agency, much like Visa or Mastercard. An agent remains in good standing with the credit agency by selling goods to whoever wants them. Being in good standing implies that one can purchase goods whenever the need arises. The agent is excluded from all future trade if he or she deviates.

Note that credit is supported in spite of the fact that, given the continuum of agents, no two agents will ever meet more than once. Agents are spatially separated, but perfect recordkeeping implies that credit works perfectly.

4 Imperfect Memory: A Monetary Economy

Now assume, at the other extreme, that there is no recordkeeping. When two agents meet, from each agent's point of view the other is anonymous - histories are private information.

We confine the analysis here to stationary equilibria, i.e. equilibria where agents' trading strategies and the quantity of money in circulation are constant for all t. In such an equilibrium, all agents are holding either one unit of money or nothing, at the end of each period. Given symmetry, it is as if there are were only two goods, and we let V_g denote the value of holding nothing (and thus being able to produce), and V_m the value of holding money at the end of the period. The fraction of agents holding money is μ, and the fraction holding nothing is $1-\mu$. If two agents who can produce meet, they will trade only if there is a double coincidence of wants, which occurs with probability x^2. If two agents with money meet, they may trade or not, since both are indifferent, but in either case they each end the period holding money. If two agents meet and one has money while the other can produce, the agent with money will want to trade if the other agent can produce a good she consumes, but the agent who can produce may or may not want to accept money.

From an individual agent's point of view, let π denote the probability that other agents accept money, where $0 \leq \pi \leq 1$, and let π' denote the probability with which the agent accepts money. Then, we can write the Bellman equations as

$$V_g = \frac{1}{1+r}\left\{ \begin{array}{c} \mu x \max_{\pi' \in [0,1]} \left[\pi'\left(V_m - \gamma\right) + (1-\pi')V_g\right] + \mu(1-x)V_g \\ +(1-\mu)\left[x^2(u^* - \gamma + V_g) + (1-x^2)V_g\right] \end{array} \right\}, \quad (32)$$

$$V_m = \frac{1}{1+r}\left\{\mu V_m + (1-\mu)\left[x\pi(u^* + V_g) + (1-x\pi)V_m\right]\right\}. \quad (33)$$

In (32), an agent with nothing at the end of the current period meets an agent with money next period with probability μ. The money-holder will want to trade with probability x, and if the money-holder wishes to trade, the agent chooses the trading probability π' to maximize end-of-period value. With probability $1-\mu$ the agent meets another agent who can produce, and trade takes place with probability x^2. If the agent trades, she consumes and then is holding nothing again.

Similarly, in (33), an agent holding money meets another agent holding money with probability μ, and meets an agent who can produce with probability $1-\mu$. Trade with a producer occurs with probability $x\pi$, as the money-holder likes the producer's good with probability x, and the producer accepts money with probability π.

It is convenient to simplify the Bellman equations, as we did in the previous two sections, by manipulating (32) and (33) to get

$$rV_g = \mu x \max_{\pi' \in [0,1]} \pi'(V_m - V_g - \gamma) + (1-\mu)x^2\left(u^* - \gamma\right), \quad (34)$$

$$rV_m = (1-\mu)x\pi(u^* + V_g - V_m). \quad (35)$$

Now, there are potentially three types of stationary equilibria. One type has $\pi = 0$, one has $0 < \pi < 1$, and one has $\pi = 1$. The first we can think of as a non-monetary equilibrium (money is not accepted by anyone), and the latter two are monetary equilibria. Suppose first that $\pi = 0$. Then, an agent holding money would never get to consume, and anyone holding money at the first date would throw it away in order to facilitate production, so we have $\mu = 0$. Then, from (34), the expected utility of each agent in equilibrium is

$$W_0 = (1+r)V_g = \frac{(1+r)x^2 (u^* - \gamma)}{r}. \tag{36}$$

Next, consider the mixed strategy equilibrium where $0 < \pi < 1$. In this symmetric equilibrium we have $\pi' = \pi$, so for the mixed strategy to be optimal, from (34) we must have $V_m = V_g + \gamma$. From (34) and (35), we can then solve for π to get

$$\pi = \frac{(1-\mu)x^2 (u^* - \gamma) + r\gamma}{(1-\mu)x (u^* - \gamma)} \tag{37}$$

For this equilibrium to exist, the solution in (37) must give $\pi < 1$, so

$$(1-\mu)x(1-x) (u^* - \gamma) - r\gamma > 0 \tag{38}$$

is required for existence. Note that (38) is a stronger condition than (31). Then, in the mixed-strategy equilibrium,

$$V_m - \gamma = V_g = \frac{(1-\mu)x^2 (u^* - \gamma)}{r}. \tag{39}$$

Now, since $V_m > V_g$ in the mixed-strategy equilibrium, all agents who are initially endowed with money will wish to hold it rather than throwing it away. Therefore $\mu = M$ in equilibrium. Further, if we let our welfare measure be the expected utility of the representative agent, before agents receive their money endowments at $t = 0$, then welfare in the mixed-strategy equilibrium is

$$W_\Phi = (1+r)\left[MV_m + (1-M)V_g\right] = \frac{(1+r)[(1-M)x^2 (u^* - \gamma) + Mr\gamma]}{r} \tag{40}$$

note, from (36) and (39), that an agent holding nothing in the mixed strategy monetary equilibrium is worse off than in the non-monetary equilibrium. However, given (38), an agent holding money in the mixed strategy equilibrium is better off than agents in the non-monetary equilibrium. From (36) and (40), welfare may be higher or lower in the mixed strategy equilibrium than in the non-monetary equilibrium. We have $W_\Phi - W_0 > 0$ if and only if

$$-x^2(u^* - \gamma) + r\gamma > 0,$$

and $W_\Phi - W_0 < 0$ if the inequality is reversed.

Next, consider the equilibrium where $\pi = 1$. Here, it must be optimal for the producer to choose $\pi' = \pi = 1$, so we must have $V_m \geq V_g + \gamma$. Conjecturing that this is so, from (34) and (35) we obtain

$$V_m - V_g = \frac{x\left\{(1-\mu)(1-x)u^* + \left[\mu + (1-\mu)x\right]\gamma\right\}}{r+x} \tag{41}$$

Therefore for the conjecture that $\pi' = 1$ is a best response to $\pi = 1$ to be correct, we require $V_m - V_g \geq \gamma$ which, from (41), gives

$$(1-\mu)x(1-x)(u^* - \gamma) - r\gamma \geq 0 \tag{42}$$

which is identical to (38), for the mixed strategy equilibrium, except that (42) is a weak inequality. Further, we will have $\mu = M$, as all agents with a money endowment will strictly prefer holding money to throwing it away and holding nothing ($V_g < V_m$). Now, using (34), (35), and (41), calculate welfare in the equilibrium with $\pi = 1$ as for the mixed strategy equilibrium to get

$$W_1 = \frac{(1+r)(1-M)x(u^* - \gamma)}{r}[M + (1-M)x]. \tag{43}$$

The equilibrium with $\pi = 1$ exists for any $M \in (0,1)$ if and only if (38) holds. Note that W_1 is a quadratic function of M, with $W_1 = W_0$ for $M = 0$ and $W_1 = 0$ for $M = 1$. Further $\frac{\partial W_1}{\partial M} > 0$ for $M = 0$ if and only if $x < \frac{1}{2}$, and if $x \geq \frac{1}{2}$, then W_1 is strictly decreasing in M for all $M \in (0,1)$. Now, if the mixed strategy equilibrium exists, then from (40) and (43), we have

$$W_1 - W_\Phi = \frac{(1+r)M}{r}[(1-M)x(1-x)(u^* - \gamma) - r\gamma] > 0,$$

given (38). So, if the mixed strategy equilibrium exists, it is dominated in welfare terms by the equilibrium with $\pi = 1$.

To summarize:

1. If $x < \frac{1}{2}$, and $\gamma < \frac{x(1-x)(u^*-\gamma)}{r}$, then
   (a) For $M \in (0, \frac{x(1-x)(u^*-\gamma)-r\gamma}{x(1-x)(u^*-\gamma)})$, all three equilibria exist and $W_0 < W_\Phi < W_1$ for M sufficiently small.
   (b) For $M \in [\frac{x(1-x)(u^*-\gamma)-r\gamma}{x(1-x)(u^*-\gamma)}, 1]$, only the $\pi = 0$ equilibrium exists.
2. If $x \geq \frac{1}{2}$, and $\gamma < \frac{x(1-x)(u^*-\gamma)}{r}$, then
   (a) For $M \in (0, \frac{x(1-x)(u^*-\gamma)-r\gamma}{x(1-x)(u^*-\gamma)})$, all three equilibria exist and $W_0 > W_1 > W_\Phi$.
   (b) For $M \in [\frac{x(1-x)(u^*-\gamma)-r\gamma}{x(1-x)(u^*-\gamma)}, 1]$, only the $\pi = 0$ equilibrium exists.

Note that, in equilibrium in the imperfect memory economy, welfare is lower in all circumstances than in the case with perfect memory, from (28), (36), (40), and (43). Kocherlakota (1998) shows a number of examples that illustrate how, in general, money can do no better than standing in for memory. In any economy, monetary exchange can do no more than what is accomplished by perfect recordkeeping. Money is memory in the sense that it keeps a record of socially desirable actions (in this economy producing for others), and one is rewarded for holding money, and thus for carrying out socially desirable actions.

However, in this economy, monetary exchange may do no better than barter exchange. Indeed, there are cases where money is valued, but it makes agents worse off than they would be with barter. We obtain this result due to the unrealistic assumption that money takes up space. For technical convenience, it is assumed in this model that an agent who is holding money cannot produce, and agents cannot hold more than one unit of money. However, this drives some of the results. Money has beneficial effects in that an agent holding money can trade in single coincidence meetings, where this was not possible with barter exchange. However, money has negative effects, in that having more agents holding money can reduce exchange. For example, when two agents with money meet, they do not trade, in spite of the fact that a double coincidence of wants may have occurred. The two agents could trade if they threw their money away, but they will not do it in equilibrium.

In this economy, it requires a sufficiently severe double coincidence problem ($x < \frac{1}{2}$) for monetary exchange to do any good. Further, note that, if $x = 1$ then barter exchange in the imperfect memory economy yields the same allocation as in the perfect memory economy. Memory is irrelevant in this case, as there are no frictions in barter exchange. This illustrates a more general point. Two necessary conditions for money to be socially useful are that there be imperfect recordkeeping and a double coincidence problem. The first is a friction that makes exchange using credit difficult, and the second is a friction that makes barter exchange difficult.

This basic search model of money provides a nice formalization of the absence-of-double-coincidence friction discussed by Jevons. The model has been extended to allow for divisible commodities (Trejos and Wright 1995, Shi 1995), and a role for money arising from informational frictions (Williamson and Wright 1994). Further, it has been used to address historical questions (Wallace and Zhou 1997, Velde, Weber and Wright 1999).

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6 Monetary Exchange with Divisibilities: Lagos-Wright

The basic monetary search model with indivisible money and goods gives us some insight into the key frictions that make monetary exchange useful. However, there are many issues that the model is not equipped to deal with. For example, it seems difficult to use the model to understand monetary policy or the effects of inflation. We might imagine relaxing some of the assumptions in the last section in order to address these and other issues. For example, we could simply assume that money and goods are perfectly divisible, and otherwise work with the same model. This does not help much, as now the model is intractable. The key problem is that we will have to track over time the distribution of money balances across the population. While we might imagine computing equilibria in such a model, the state object in this economy is a distribution, and therefore not generally amenable to analysis.

One approach to incorporating indivisibilities is to use a representative household device, along the lines of Lucas (1992). This is the approach taken, for example, by Shi (1997). In these models a household consists of multiple agents, who perform different tasks during the period, i.e. the agents in the household buy goods, sell goods, and may buy and sell assets as well. All agents in the household are together at the beginning of the period, and they unite again at the end of the period, when they pool their assets. Thus, there is no issue of having to deal with how assets are distributed across the population, since the representative household must hold the economy's supplies of assets in equilibrium.

A second approach, which is perhaps more tractable and versatile than the Lucas-Shi multi-agent household approach, was developed by Lagos and Wright (2005). This approach relies on quasilinear preferences and alternating periods of centralized trading to deal in a simple way with the distribution of money and other assets across the population. We will follow this approach in this section.

What we will construct here, using a Lagos-Wright approach, is a tractable model of money which can be extended to include credit, banking, alternative means of payment, and other assets. There of course exist other standard models of money in the literature which have been used widely, and which are also tractable. Two widely-used models are the cash-in-advance model and the money-in-the-utility-function model. The case for our approach is that the roles for money and other assets in these models are derived from features of the environment, rather than by assumption. Our approach is therefore immune to the Lucas critique (Lucas 1976) - private decision rules are in general invariant to changes in policy rules. For more details on why "deep" models of money are desirable, see Kareken and Wallace (1980), Wallace (1998), and Williamson and Wright (2009).

6.1 The Model

Time in the model is indexed by $t = 0, 1, 2, 3, ...$, and there are two subperiods, which we will denote day and night. Calling the subperiods day and night has no significance - these are just names to help us remember which subperiod comes first. The population consists of a continuum of *buyers* and a continuum of *sellers*, each having unit mass. Each buyer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(x_t) - H_t]$$

Here, $0 < \beta < 1$, x_t is consumption in the day, and H_t is labor supply at night. Assume that $u(\cdot)$ is strictly concave and twice continuously differentiable with $u'(0) = \infty$ and $u(0) = 0$. There exists some $\hat{x} > 0$ such that $u(\hat{x}) - \hat{x} = 0$, and define q^* as the solution to $u'(q^*) = 1$. Each seller has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [-h_t + X_t],$$

where h_t is labor supply in the day and X_t is consumption at night. Sellers can produce only in the day, and buyers can produce only at night. When productive, an agent can produce one unit of the perishable consumption good with each unit input of labor.

During the day, a buyer has a probability α of being randomly matched with a seller. Thus, a fraction $1 - \alpha$ of buyers, and a fraction $1 - \alpha$ of sellers, do not trade during the day. Buyers do not learn whether they will trade during the day until the beginning of that day. During the night, all agents meet in a competitive market, and trade at competitive prices.

6.2 Perfect Memory

We first assume that each agent in each period, and in each subperiod, has access to a complete records of the histories of all other agents, including who they have met, and what they have produced and consumed. This will provide us with a useful benchmark, just as in the case with indivisibilities. We can work out what allocations are optimal under perfect memory, and then compare these to the equilibrium allocations we obtain with perfect memory and credit arrangements and under imperfect memory with monetary exchange.

In deriving optimal allocations, we will confine attention to allocations where any agents not matched during the day of a subperiod receive zero consumption and do not produce in the following night. Matched buyers will all receive the same consumption across periods, as will matched sellers. Since it will be suboptimal to throw goods away, let x denote the consumption of matched buyers during the day, which is equal to the production of matched sellers. Similarly, let X denote consumption of sellers during the night who were matched in the day, which is equal to the production of each buyer during the night who was matched in the day. Thus, the period utility of a matched buyer is $u(x) - X$, and the period utility of a matched seller is $-x + X$. All other agents receive zero utility during the period.

6.2.1 Optimality

To map out optimal allocations, we will choose (x, X) so as to maximize the lifetime utility of buyers, holding constant the lifetime utility of sellers, and subject to incentive constraints. First, let k denote the period utility of matched sellers. One constraint in the social planner's problem will then be

$$-x + X \geq k. \tag{44}$$

For sellers to be willing to produce when matched, we then require

$$k \geq 0. \tag{45}$$

For matched buyers, production during the night of period t is incentive compatible if and only if

$$-X + \alpha \sum_{s=t+1} \beta^s \left[u(x) - X\right] \geq 0,$$

which we can rewrite as

$$X \leq \psi u(x), \tag{46}$$

where

$$\psi = \frac{\alpha\beta}{1-(1-\alpha)\beta}$$

Here, we are assuming, for buyers and sellers, that the harshest punishment available to the social planner is to deny an agent consumption for the duration of his or her life, if he or she deviates from the plan. Note that ψ is increasing in both α and β. That is, as the frequency of trade during the day increases, and as the discount factor increases, the reward for behaving according to the plan rises for the buyer, relative to defecting.

To determine the set of optimal allocations, first choose k, then maximize $u(x) - X$ subject to (44) and (46). Clearly, (44) always binds as, if not, we could reduce X, increase the value of the objective function, and relax (46). First, suppose that (46) does not bind. Then, the optimal allocation would be $(q^*, q^* + k)$. For this allocation to satisfy (46) requires that

$$k \leq \psi u(q^*) - q^*. \tag{47}$$

If $k > k^*$ then (47) is violated, where

$$k^* = \psi u(q^*) - q^*.$$

In this case (46) must bind if there is a solution, and x solves

$$x + k = \psi u(x). \tag{48}$$

Let $\hat{x}$ denote the solution to

$$\psi u'(\hat{x}) = 1,$$

and define

$$\hat{k} = \psi u(\hat{x}) - \hat{x}.$$

Note that, since $\hat{x} < q^*$, that $\hat{k} > k^*$.

The optimal allocation is then the following. If $k^* < 0$, then for $k \in [0, \hat{k}]$, x solves (48) and $X = x + k$. If $k^* \geq 0$, then for $k \in [0, k^*]$, $x = q^*$ and $X = q^* + k$, and for $k \in [k^*, \hat{k}]$, x solves (48) and $X = x + k$. For $k > \hat{k}$ there is no feasible optimal allocation.

6.2.2 Credit Equilibrium

With perfect memory, the only trades will be intertemporal exchanges between a matched buyer and seller. They meet in the day, when the seller produces x for the buyer, and then meet again in the night, when the buyer produces X for the seller. We will confine attention to an equilibrium where default (the buyer does not produce X as agreed during the night) triggers an off-equilibrium punishment whereby no one trades with the defaulter, and he or she attains zero continuation utility.

When a buyer and seller meet during the day, they bargain over (x, X). We will assume Nash bargaining, with θ denoting the bargaining weight of the buyer. Then, (x, X) solves

$$\max_{x,X} [u(x) - X]^{\theta} [-x + X]^{1-\theta}$$

subject to

$$-X + v \geq 0 \tag{49}$$

where v is the continuation value of the buyer after he or she repays his or her debts in the night. In the problem above, v is taken as given, but in equilibrium v is determined by

$$v = \frac{\alpha\beta [u(x) - X]}{1 - \beta} \tag{50}$$

Now, if the constraint (49) does not bind, then $x = q^*$, as this maximizes total surplus, and

$$X = (1 - \theta)u(q^*) + \theta q^*$$

Then, from (50), if (49) does not bind in equilibrium, then

$$v = \frac{\alpha\beta\theta [u(q^*) - q^*]}{1 - \beta},$$

so for (49) to hold, we require

$$q^* \leq \frac{\alpha\beta\theta u(q^*)}{1 - \beta(1 - \alpha\theta)}. \tag{51}$$

Thus, (51) is a necessary and sufficient condition for existence of the equilibrium $(x, X) = [q^*, (1-\theta)u(q^*) + \theta q^*]$, where the incentive constraint for the buyer does not bind.

Now, consider the case where the constraint (49) binds. Then, solving the Nash bargaining problem, we get the first-order condition

$$\theta u'(x)[-x+v] - (1-\theta)[u(x)-v] = 0,$$

which we can write as

$$v = \frac{\theta x u'(x) + (1-\theta)u(x)}{\theta u'(x) + 1 - \theta} \equiv z(x). \tag{52}$$

In equilibrium, (50) holds, so

$$v = \frac{\alpha\beta\,[u(x)-v]}{1-\beta},$$

which we can write as

$$v = \frac{\alpha\beta u(x)}{1-\beta(1-\alpha)}. \tag{53}$$

Then, an equilibrium where the buyer's incentive constraint binds is (v, x) solving (52) and (53), with $x < q^*$. We can show that $z(0) = u(0) = 0$, so there always exists an equilibrium with $v = x = 0$. In this equilibrium, there is no trade. No seller is willing to lend to a buyer, as the seller anticipates that no one will lend to the buyer in the future. Thus, the buyer would default on any loan extended to him, and is therefore not creditworthy.

It is straightforward to show that $z'(x) > 0$ for $x \in [0, q^*]$. Now, the problem will be nicely-behaved if $z(\cdot)$ it would help if $z(\cdot)$ is convex with $z'(0)$ finite. We could find sufficient conditions on $u(\cdot)$ for these properties to hold, but simply assume these properties for now. Then, a necessary and sufficient condition for the existence of an equilibrium where the incentive constraint binds for the buyer, and $v > 0$ is

$$q^* > \frac{\alpha\beta\theta u(q^*)}{1-\beta(1-\alpha\theta)} \tag{54}$$

Then, from (51) and (54), there will then be two equilibria. One has $x = v = 0$, one has $x > 0$ and $v > 0$. In the equlilibrium with $v > 0$, either the incentive constraint binds for the buyer or it does not and $x = q^*$. There is a strategic complementarity in this model that gives rise to multiple credit equilibria. That is, the fact that other sellers are willing to lend in the future makes a given seller more willing to lend in the present.

Note that credit equilibria are in general inefficient. In particular, suppose that $k^* > 0$ so that $\psi u(q^*) - q^* > 0$ and there exist efficient allocations where the incentive constraint does not bind for the buyer. Then, there always exists θ sufficiently small that (51) does not hold, and this allocation cannot be supported as a credit equilibrium. In general, there exists a holdup problem, whereby the buyer essentially is held up for his or her reputation. The holdup problem becomes more severe as the buyer's bargaining weight, θ, falls. The seller's ability to extract surplus from the buyer acts to increase the buyer's incentive to default in equilibrium, as he or she has less to lose from defaulting.

As an example, consider the case where $\theta = 1$, and the buyer has all the bargaining power. Essentially, in the daytime a buyer matched with a seller makes a take-it-or-leave-it offer. Then, from (51), an equilibrium where the buyer's incentive constraint does not bind exists if and only if

$$q^* \leq \frac{\alpha\beta u(q^*)}{1-\beta(1-\alpha)}.$$

and this equilibrium is efficient. From (52) and (53), in an equilibrium where the buyer's incentive constraint does not bind, x solves

$$x = \frac{\alpha\beta u(x)}{1-\beta(1-\alpha)}.$$

One solution is $x = 0$ (the no-trade equilibrium), and a necessary and sufficient condition for existence of an equilibrium with $x > 0$ of this type is

$$q^* > \frac{\alpha\beta u(q^*)}{1-\beta(1-\alpha)}.$$

Thus, with $\theta = 1$ there is an equilibrium with $x = v = 0$, and one other equilibrium with $x > 0$ and $v > 0$, in which the credit constraint may or may not bind. The equilibrium with $x > 0$ and $v > 0$ is always efficient.

Another very simple example is the case where $\theta = 0$. Here, the only equilibrium is $x = v = 0$ as, if the seller takes all the surplus in trade, then the buyer would default on any loan, since he or she always has a continuation value of zero.

6.3 Imperfect Memory and Monetary Exchange

In this subsection we will shut down recordkeeping entirely in this model. This is an extreme assumption, as it eliminates exchange using credit entirely, but this will highlight the role of monetary exchange. It is straightforward to extend the model to allow for intermediate cases that permit exchange using both money and credit.

Assume that, when a buyer and seller meet during the day, that neither can observe the other's history. We will assume, however, that if either agent holds money, then money balances are observable. As well, when agents meet during the night, they can only observe prices in Walrasian markets, and not the actions of other agents.

We will assume that there is a perfectly divisible and durable object, fiat money, that can be produced only by the government. At the beginning of period 0, each buyer is endowed with M_0 units of money. The government can augment the money supply each period by making transfers to buyers in equal amounts during the night, where τ_t denotes the nominal transfer to buyers in the night of period t. The government's budget constraint is then

$$M_{t+1} = M_t + \tau_t \tag{55}$$

for all t. Further, the government sets transfers each period so that the money supply grows at a constant rate, with μ denoting the constant money growth factor, or

$$\tau_t = (\mu - 1)M_t. \tag{56}$$

6.3.1 Centralized Market: Nighttime Exchange

During the night, trade takes place on a Walrasian market, where money is exchanged for consumption goods. Let ϕ_t denote the price of money in terms of consumption goods. Typically, in monetary models, it can be more straightforward to make the numeraire consumption goods rather than money, as there is always an equilibrium where money is not valued, i.e. $\phi_t = 0$. The price level (the price of goods in terms of money) is $\frac{1}{\phi_t}$.

Note that, in equilibrium, we must have

$$\beta\phi_{t+1} \leq \phi_t, \tag{57}$$

for all t, as otherwise a buyer could increase utility by increasing labor supply in period t, hold money over until the night of period $t+1$, and reduce labor supply in period $t+1$. Further, if (57) does not hold, then a seller wishes to give up consumption in period t, hold money until period $t+1$, and increase period $t+1$ consumption. Thus, buyers wish to supply a positive quantity of labor in the night of period t, while sellers wish not to consume in the night of period t, which cannot be case in equilibrium, as all goods produced by buyers must be consumed by sellers in each night. Later, we will also show that if (57) does not hold then the nominal interest rate is negative, which represents an arbitrage opportunity. Given (57), if any sellers are holding money during the night, it is optimal for them to sell it. The demand for money during the night then comes only from buyers, who need the money to make transactions during the upcoming day. A buyer's budget constraint in the night is

$$\frac{\phi_t m_{t+1}^d}{\phi_{t+1}} = m_t^n + H_t + \phi_t\tau_t, \tag{58}$$

where m_t^d denotes the real value of money balances held at the beginning of the day in period t, and m_t^n similarly denotes real money balances at the beginning of the night.

Let $W(m)$ denote the nighttime value function, and $V(m)$ the daytime value function. Then, we can write the nighttime problem of a buyer as a dynamic programming problem, i.e.

$$W(m_t^n) = \max_{m_{t+1}^d, H_t} [-H_t + \beta V(m_{t+1}^d)], \tag{59}$$

subject to (58). Now, conjecture that the nonnegativity constraint on labor supply does not bind (to be verified later) for the buyer in the night, and substitute for H_t in (59) using (58) to get

$$W(m_t^n) = m_t^n + \phi_t \tau_t + \max_{m_{t+1}^d} \left[-\frac{\phi_t m_{t+1}^d}{\phi_{t+1}} + \beta V(m_{t+1}^d) \right]. \tag{60}$$

Two important results follow from (60). The first is that $W(m)$ is linear in m. The second is that, provided $V(\cdot)$ is well-behaved, all buyers will choose to hold the same quantity of money balances at the end of every night (and therefore at the beginning of every day). That is, in (60), the choice of m_{t+1}^d is independent of $m_t^n + \phi_t \tau_t$.

6.3.2 Decentralized Trading: Daytime Exchange

Consider a buyer and seller who meet during the day, with the buyer holding m units of real money balances (valued in terms of the price of money in the following night). Let d denote the quantity of real balances that are exchanged by the buyer for x units of goods, to be produced by the seller. Given (60), the surplus the buyer receives from trading is $u(x) - d$, and the seller's surplus is $-x + d$, since the seller's utility is linear in consumption and he or she will sell any money balances acquired in the day during the next night. Then assuming, as in our analysis of credit, that the buyer and seller engage in Nash bargaining, x and d solve

$$\max_{x,d} \left[u(x) - d\right]^{\theta} \left[-x + d\right]^{1-\theta}$$

subject to

$$d \leq m \tag{61}$$

Now, in equilibrium the constraint (61) will always bind, given (57) (except possibly if (57) holds with equality, but that will not be a problem for our purposes), so we will simply assume $d = m$ at this stage. Solving the Nash bargaining problem, we then get the following first-order condition:

$$\theta u'(x)[m - x] - (1 - \theta)[u(x) - m] = 0$$

or, solving for m in terms of x, we get

$$m = \frac{\theta x u'(x) + (1 - \theta) u(x)}{\theta u'(x) + 1 - \theta} = z(x). \tag{62}$$

Here $z(x)$ is the quantity of real balances that the buyer needs in the day to purchase x units of goods (and $z(\cdot)$ is the same function we derived in the credit problem). Then, we can rewrite the Bellman equation (60) as

$$W(m_t^n) = m_t^n + \phi_t \tau_t + \max_{x_{t+1}} \left[-\frac{\phi_t z(x_{t+1})}{\phi_{t+1}} + \beta \alpha u(x_{t+1}) + \beta (1 - \alpha) z(x_{t+1}) \right],$$

and the first-order condition for an optimum is

$$\left[-\frac{\phi_t}{\phi_{t+1}}+\beta(1-\alpha)\right]z'(x_{t+1})+\beta\alpha u'(x_{t+1})=0. \tag{63}$$

From (62), we have

$$z'(x)=\frac{u'(x)}{\theta u'(x)+1-\theta}-\frac{\theta(1-\theta)[u(x)-x]u''(x)}{[\theta u'(x)+1-\theta]^2} \tag{64}$$

Now assume, as in our analysis of the credit equilibrium with perfect memory, that $z(\cdot)$ is a convex function with $z'(0)$ finite. As we mentioned above, one can find weak conditions on $u(\cdot)$ that will guarantee these properties. This will give an interior solution to the buyer's nighttime optimization problem.

6.3.3 Equilibrium

We will look for an equilibrium where all real variables are constant over time, i.e. $x_t = x$, $X_t = X$, $m_t^d = m^d$, and $m_t^n = m^n$ for all t. In equilibrium, buyers must hold the entire stock of money at the beginning of the day, so

$$\phi_t M_{t-1} = m_t^d \tag{65}$$

for all t, i.e. money supply equals money demand. Then, (65), (55) and (56) imply that

$$\frac{\phi_t}{\phi_{t+1}}=\frac{m^d M_t}{m^d M_{t-1}}=\mu \tag{66}$$

Then, to solve for x, from (63), (64), and (66), we obtain

$$\frac{1}{\theta u'(x)+1-\theta}-\frac{\theta(1-\theta)[u(x)-x]u''(x)}{[\theta u'(x)+1-\theta]^2\,u'(x)}=\frac{\beta\alpha}{\beta\alpha+\mu-\beta} \tag{67}$$

Now, from (57) and (66), $\mu \geq \beta$ for an equilibrium to exist, which implies that the right-hand side of (67) is smaller than one in equilibrium. This then implies that $u'(x) > 1$ in equilibrium, so $x < q^*$, as long as $0 < \theta < 1$. Here, a holdup problem implies that the quantity consumed by the buyer in the day is always less than the surplus-maximizing quantity.

Now, convexity of $z(\cdot)$ implies that the left-hand side of equation (67) is increasing in x, so an increase in μ implies a decrease in x, and therefore a decrease in $z(x)$, the real quantity of money held at the beginning of the day. Higher μ implies a higher inflation rate, which erodes the purchasing power of money acquired during the night. Buyers then economize on money balances, and consumption in the day falls. The quantity of consumption for sellers is $X = z(x)$, which is also decreasing in μ.

Now, one problem we have not addressed yet is that the government faces a limited commitment problem in collecting taxes. To be consistent with the assumption of lack of commitment in our environment, it should be the case that buyers cannot commit to paying their taxes to the government. If $\mu \geq 1$, this is not an issue, of course, since buyers will be pleased to receive positive transfers from the government. However, if $\mu < 1$, then buyers are taxed each period, and it must be incentive compatible in equilibrium for buyers to pay these taxes.

To determine what is incentive compatible taxation, we have to first ask what punishments it is possible for the government to impose on agents who default on their tax liabilities. There are no prisons in our model, and since there is no recordkeeping there is no possibility of carrying out individual-specific punishments. However, the government has the ability to determine the money growth rate. The worst punishment that the government can impose is to set $\mu = \infty$ if any buyer defaults on their tax liabilities. This is a global punishment which is essentially equivalent to universal autarky. If $\mu \to \infty$, then real money balances go to zero in the limit, and consumption is zero for buyers and sellers in the day and night, respectively. This punishment is optimal, as it will never be carried out in equilibrium, and is as effective as possible in enforcing good behavior. It is important that the government have the ability to commit to this policy since, once a defection occurs, the government would not want to carry out the punishment.

In equilibrium, during any night a fraction $1 - \alpha$ of buyers will not have traded in the previous day, and will be holding M_t units of money. These buyers will then pay a tax of $(1 - \mu)M_t$ units of money to the government, and will work zero hours, as they will then be holding the quantity of money that they wish to take into the next day. For these agents, not paying their taxes has no value. They cannot reduce labor supply, which is already zero, they cannot exchange the extra money for goods, and money held over until the next period will have zero value. Now, consider the fraction α of buyers who traded in the previous period. These agents each have zero units of money balances, and in equilibrium each would work enough hours to acquire M_t units of money balances. Since these buyers are the only ones working, in equilibrium they would be supplying $X = z(x)$ units of labor. If one of these buyers defects by defaulting on their taxes, they would then choose to work zero hours in the current night, and would face autarky thereafter. Thus, the incentive constraint for these agents is

$$-z(x) + \frac{\alpha\beta[u(x) - z(x)]}{1 - \beta} \geq 0,$$

and given (62), we can write this as

$$\theta u'(x)\left\{\alpha\beta u(x) - [1 - \beta(1 - \alpha)]\, x\right\} - (1 - \theta)(1 - \beta)u(x) \geq 0 \qquad (68)$$

The incentive constraint (68) will in general constrain μ.

Note how quasilinearity in the buyer's utility function helps to simplify matters. We have already shown that, at the end of the night, each buyer will choose to hold the same quantity of money balances. Therefore, at the end of period t, each buyer holds M_t units of money. The fraction α of buyers who meet sellers will trade away all of their money in equilibrium for goods, while the $1-\alpha$ buyers who do not meet sellers simply carry their money into the next night. Therefore, at the beginning of every night, there are α buyers with zero money balances, $1-\alpha$ buyers with M_t units of money, α sellers with M_t units of money, and $1-\alpha$ sellers with with zero units of money. The α buyers who have no money work a positive amount so that they can acquire the money held by sellers, then all buyers receive a transfer from the government, at which point all buyers have M_{t+1} units of money. Note that the buyers who begin the night with M_t units of money work zero hours. Quasilinear preferences guarantee that, whenever agents meet in the centralized market, the distribution of money balances across the population is optimally "reset" in a manner exogenously determined by the outstanding stock of money.

The final step in characterizing a monetary equilibrium is to determine asset prices. This is quite straightforward, as long as none of the assets in question can be traded during the day. Here, we will assume that any assets other than money can be counterfeited at zero cost, and therefore will not be accepted in exchanges during the day. For our purposes, we wish to price two assets, a real bond and a nominal bond.

First, a one-period real bond is an asset that is traded among private agents, is issued in the night of period t, and pays off one unit of consumption goods in the night of period $t+1$. This asset trades during the current night for s_t units of consumption goods. Given the preferences of buyers and sellers, we must have

$$s_t = \beta$$

in equilibrium, which implies that the real interest rate is $\frac{1}{\beta}-1$, the rate of time preference. Similarly, the payoff on a one-period nominal bond is one unit of money in the night of period $t+1$, and the bond sells for w_t units of money in the night of period t. In equilibrium, we must have

$$\phi_t w_t = \beta \phi_{t+1},$$

so in equilibrium we have

$$w_t = \frac{\beta \phi_{t+1}}{\phi_t} = \frac{\beta}{\mu},$$

and so the equilibrium nominal interest rate is $\frac{\mu}{\beta} - 1$. There is a standard Fisher effect here. The real interest rate is unaffected by changes in the money growth rate, and the nominal interest rate increases roughly one-for-one with increases in the money growth rate, which is equal to the inflation rate (from 66). Note that the restriction $\mu \geq \beta$, required for an equilibrium with valued money to exist, implies that the nominal interest rate must be nonnegative in equilibrium. If the nominal interest rate were negative, then an agent could make infinite profits by borrowing at the nominal interest rate in the current night and holding money until the next night.

Note that money is *neutral* in this model, in the sense that equilibrium quantities do not depend on the level of the money supply, i.e. no real quantities depend on M_0 in equilibrium. From (65),

$$\phi_t = \frac{m^n}{M_{t+1}} = \frac{m^n}{M_0 \mu^{t+1}}$$

Therefore, since m^n is unaffected by changes in M_0, holding μ constant, then ϕ_t increases in proportion to $\frac{1}{M_0}$. That is, the price level is proportional to M_0. Money is not *superneutral*, however, as changes in the money growth rate matter for real quantities.

6.3.4 Optimality

If we take as our welfare criterion the sum of expected utilities across buyers and sellers in equilibrium, welfare is then proportional to

$$W = u(x) - x,$$

which is the surplus from trade between a buyer and seller who meet in the night. Note that consumption and production of buyers and sellers in the night cancel out in the welfare function. Since $x < q^*$ in equilibrium and $u(x) - x$ is increasing in x for $x < q^*$, the policy objective is to set μ to make x as large as possible, subject to the constraint (68), and $\mu \geq \beta$. We have already determined that x is decreasing in μ, so an optimal policy minimizes μ subject to (68), and $\mu \geq \beta$.

If (68) does not bind at the optimum, then the optimal money growth factor is $\mu = \beta$. This is a *Friedman rule* (see Friedman 1969), which is generally defined as a monetary rule that implies a zero nominal interest rate in all states of the world. Note that the nominal interest rate here is $\frac{\mu}{\beta} - 1$, which is zero at the Friedman rule, and strictly positive otherwise. Note however that, even if the Friedman rule is feasible, it does not in general achieve efficiency. The right-hand side of (67) is greater than or equal to 1, given $\mu \geq \beta$. However, so long as $0 < \theta < 1$, (67) implies that $x < q^*$, even if $\mu = \beta$. This follows from the fact that there are two inefficiencies here. The first is a standard monetary inefficiency, which is corrected in most monetary models by a Friedman rule. This inefficiency results from the tendency of agents to economize too much on money balances than is socially optimal, as they discount the future. With only this inefficiency in the model, the nominal interest rate is a measure of the extent of the inefficiency, and driving the nominal interest rate to zero acts to equate the rates of return on all assets, including money. Here, however, there is a second source of inefficiency, which is the holdup problem that occurs in trade during the day. In a meeting between a buyer and a seller, the buyer's cost of accumulating money balances is sunk, and cannot therefore enter the bargaining problem. The seller then has an incentive to extract surplus from the buyer, and this is socially inefficient. Monetary policy cannot in general correct both distortions - the distortion from the holdup problem still exists at the Friedman rule. A holdup problem sometimes also occurs in problems in industrial organization and labor economics.

6.3.5 Examples

Suppose first that $\theta = 1$. Then (65) gives

$$z(x) = x$$

and (67) becomes

$$\frac{1}{u'(x)} = \frac{\beta\alpha}{\beta\alpha + \mu - \beta}. \tag{69}$$

In this case, x is clearly decreasing in μ, and if the Friedman rule is attainable, then $\mu = \beta$ implies that $x = q^*$, which is efficient. To determine what policies are attainable, (68) gives

$$x \leq \frac{\alpha\beta u(x)}{1 - \beta(1 - \alpha)} \tag{70}$$

The Friedman rule will then be feasible (and therefore optimal) if and only if, from (70),

$$q^* \leq \frac{\alpha\beta u(q^*)}{1 - \beta(1 - \alpha)} \tag{71}$$

However, if (71) does not hold, then (70) must bind at the optimum. Then, let $\bar{x}$ be the solution to

$$\bar{x} = \frac{\alpha\beta u(\bar{x})}{1 - \beta(1 - \alpha)},$$

and the optimal money growth factor, from (69), is

$$\mu = \beta \left\{1 + \alpha \left[u'(\bar{x}) - 1\right]\right\}$$

In this case, monetary policy can always achieve an efficient allocation, even when the government cannot attain the Friedman rule. If you check our characterization of optimal allocations, you will see that the cases where the Friedman rule is not attainable are the same ones where incentive constraints bind for the social planner. The Friedman rule is not attainable in some cases because taxes are sufficiently high at the Friedman rule that buyers would want to default on their tax liabilities. Similarly, sometimes allocations that maximize the surplus in meetings during the day are not feasible for the social planner, as the planner cannot induce buyers to produce during the night.

Efficiency can always be attained in equilibrium here with $\theta = 1$, as this is the case where there is no holdup problem, and monetary policy then needs to correct only one distortion. Note that, at the optimum, money is literally memory, as we attain exactly the same equilibrium allocation as with perfect memory and credit. Of course there is always an equilibrium where $x = X = 0$ and there is no trade, either in a monetary economy with imperfect memory, or in the perfect memory economy. Indeed, there in general will be many other inefficient equilibria as well.

Now, consider the other extreme, which is $\theta = 0$. In this case, just as in the credit economy with perfect memory, the only equilibrium is one with $x = X = 0$, where $\phi_t = 0$ for all t. Since buyers never receive any surplus from trading in the day, the demand for money is zero, and so money cannot have value in equilibrium.

6.3.6 Discussion

The above illustrates some of the benefits of being explicit concerning the frictions that make money and other assets useful in exchange. A key idea is that the social role for money comes from lack of memory (imperfect recordkeeping). There are many extensions of this basic framework which allow for the coexistence of money and credit (Sanches and Williamson 2009), financial intermediation (Williamson and Wright 2009, Williamson 2009), and asset trade (Lester, Postlewaite, and Wright 2009; Williamson and Wright 2009; Lagos 2008), for example.

Another extension is to include short-run nonneutralities of money. One way to do this is through market segmentation (see for example Williamson 2006). A second approach (if you like that sort of thing) is sticky prices (see Williamson and Wright 2009).

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